REPRESENTATION THEORY

ASSIGNMENT DUE ON OCTOBER 3, 2011

Assume whenever you wish that K is a field of characteristic $> n$

(1) Assuming that $K_{\mu\lambda} > 0$ if and only if $\mu \leq \lambda$, show that $N_{\lambda\mu} \geq 2$ if $\mu' < \lambda$ [Hint: use the dual RSK correspondence].

(2) Express the product $m_{(1,1)}m_{(2)}$ as a linear combination of monomial symmetric functions.

(3) Let $X_{\lambda\mu} = \mathrm{trace}(w_\mu, V_\lambda)$, where w_μ is an element with cycle decomposition of shape μ. Show that

$$\det X = \prod_{\lambda \vdash n} \prod_{i=1}^{\infty} m_i(\lambda)!$$

[Hint: Compute the determinant of the matrix $P_{\lambda\mu} = \mathrm{trace}(w_\mu, X_\lambda)$ by showing that it is upper triangular and then computing its diagonal elements]

(4) For a partition λ, let $m_i(\lambda)$ denote the number of times i occurs in λ. Let $g_i(\lambda)$ denote the number of parts of λ which occur at least i times. For example, if $\lambda = (3, 2, 2, 2, 1, 1)$ then $m_2(\lambda) = 3$, whereas $g_2(\lambda) = 2$. Show that

$$\sum_\lambda m_i(\lambda) = \sum_\lambda g_i(\lambda).$$

(5) Show that the determinant of the character table of S_n (semisimple case) is given by

$$\det X = \prod_{\lambda \vdash n} \prod_i \lambda_i$$

the product of all the parts of all partitions of n [Hint: use the two previous problems].

(6) Let $\omega : \Lambda_K^n \to \Lambda_K^n$ be given by

$$\omega(e_\lambda) = h_\lambda \text{ for each } \lambda \vdash n.$$

Show that ω^2 is the identity map [Hint: use transition matrices].

(7) Show that the number of permutations in S_n which commute with an element with cycle decomposition of shape λ is given by

$$|Z_{S_n}(w_\lambda)| = 1^{m_1(\lambda)} m_1! 2^{m_2(\lambda)} m_2! \cdots .$$

(8) Let P_n denote the ring of polynomials in $K[x_1, \ldots, x_n]$ which are invariant under any permutation of the variables. Define $r_n : \Lambda_K \to P_n$ by

$$r_n(f)(x_1, \ldots, x_n) = f(x_1, \ldots, x_n, 0, \ldots)$$

Let Par_n denote the set of partitions of n with at most n parts. Show that $\{r_n(m_\lambda) | \lambda \in \mathrm{Par}_n\}$ is a basis of P_n. Moreover, $r_n(m_\lambda) = 0$ if $\lambda \notin \mathrm{Par}_n$.

(9) Show that $\{r_n(h_\lambda) | \lambda' \in \mathrm{Par}_n\}$ and $\{r_n(e_\lambda) | \lambda' \in \mathrm{Par}_n\}$ are bases of P_n.