REPRESENTATION THEORY

ASSIGNMENT DUE ON SEPTEMBER 5, 2011

Assume throughout that K is an algebraically closed field

(1) Let X be a finite G-space an $(\rho_X, K[X])$ denote the associated permutation representation. Show that $\mathrm{trace}(\rho_X(g); K[X]) = \#\{x \in X | g \cdot x = x\}$.

(2) Let X be the set of all partitions of $\mathbf{n}$. What are the orbits for the action of S_n on X (recall that an orbit is a minimal non-empty set that is preserved under the action of S_n)?

(3) A G-space is said to be *transitive* if it has only one orbit. If H is a subgroup of G, then G acts on the set G/H of cosets by $g \cdot (xH) = gxH$. Show that G/H is a transitive G-space.

(4) Two G spaces X and Y are said to be isomorphic of there exists a bijection $\phi : X \to Y$ which preserves the G-actions:

$$\phi(g \cdot x) = g \cdot \phi(x) \text{ for all } x \in X,\ g \in G.$$

Show that every transitive G-space is isomorphic to a coset space (note that isomorphic G-spaces will give rise to isomorphic permutation representations).

(5) Let X be a transitive G-space and $K[X]_0$ denote the subspace of $K[X]$ consisting of functions whose sum over X is 0. Then $K[X]_0$ has a complement if and only if the characteristic of K does not divide $|X|$.

(6) Assume that X is a transitive G-set and that the characteristic of K does not divide $|X|$. Show that the permutation representation is completely reducible (a sum of simple representations).

(7) Assume that X is a transitive G-set and that the characteristic of K does not divide $|X|$. Show that $K[X]_0$ (as in Problem 5) is simple if and only if, whenever (x_1, y_1) and (x_2, y_2) are two pairs of distinct elements in X, then there exists $g \in G$ such that $g \cdot x_1 = x_2$ and $g \cdot y_1 = y_2$ (in other words, the action of G on X is doubly transitive).

(8) Given three finite sets X, Y and Z, and kernels $k_1 : X \times Y \to K$ and $k_2 : Y \times Z \to K$,

$$T_{k_1} \circ T_{k_2} = T_{k_1 * k_2}$$

where $k_1 * k_2 : X \times Z \to K$ is defined by

$$k_1 * k_2(x, z) = \sum_{y \in Y} k(x, y) k(y, z).$$

(9) *(Gelfand's trick) Let K be an algebraically closed field, X be a finite G-set. Assume that the permutation representation $(\rho_X, K[X])$ is completely reducible. Prove that $K[X]$ has a multiplicity-free decomposition into simples (i.e., each simple occurs with multiplicity at most one) if for all $(x, y) \in X^2$, there exists $g \in G$ such that $y = g \cdot x$ and $x = g \cdot y$. [Hint: prove that $\mathrm{End}_G K[X]$ is a commutative algebra.]

(10) *Let K be an algebraically closed field whose order does not divide n. Let P_n denote the set of vertices of a regular n-gon in $\mathbf{R}^2$, centred at the origin. The dihedral group D_{2n} is defined to be the group of all linear transformations $\mathbf{R}^2 \to \mathbf{R}^2$ which map P_n onto itself. Determine the dimensions and multiplicities of the simple representations which occur in the permutation representation of D_{2n} on $K[P_n]$.

(11) Compute the characters of the irreducible representations V_0, V_1 and V_2 of S_5, which come from the subset permutation representations (assume that K is an algebraically closed field of characteristic > 5).

(12) Let K be an algebraically closed field of characteristic > 3 compute the character table of S_3 using partition representations as explained in class.

(13) Let K be an algebraically closed field of characteristic > 3 compute the character table of S_4 using partition representations as explained in class