REPRESENTATION THEORY

ASSIGNMENT DUE ON AUGUST 29 2011

Assume throughout that K is an algebraically closed field

(1) Let R be a semisimple K-algebra. Show that an R-module V is simple if and only if $\dim_K(\operatorname{End}_R V) = 1$. Does this fail when R is not semisimple or when K is not algebraically closed?

(2) Let R be a semisimple K-algebra. An R-module V has a multiplicity-free decomposition (meaning that it is isomorphic to a sum of pairwise non-isomorphic simple modules) if and only if its endomorphism algebra $\operatorname{End}_R V$ is commutative.

(3) If R is a finite dimensional semisimple K-algebra and V and W are finite dimensional R-modules such that

$$\dim \operatorname{End}_R V = \dim \operatorname{Hom}_R(V, W) = \dim \operatorname{End}_R W$$

then V and W are isomorphic.

(4) Let $V_1, \ldots, V_r$ be finite dimensional vector spaces over K. In the K-algebra

$$R = \bigoplus_{i=1}^{r} \operatorname{End}_K V_i$$

let ϵ_i denote the identity endomorphism id_{V_i} of V_i.

(a) Show that the centre of R (the set of all $z \in R$ such that $zr = rz$ for all $r \in R$) is spanned by $\epsilon_1, \ldots, \epsilon_r$.

(b) Show that the only primitive central idempotents in R are the elements $\epsilon_1, \ldots, \epsilon_r$.

(5) Let G be a finite group and K be any field. Show that the centre of $K[G]$ consists of the K-valued functions on G which are constant on conjugacy classes (class functions).

(6) Suppose that K is a field and G is a finite group such that the characteristic of K does not divide $|G|$. Let $V_1, \ldots, V_r$ denote a set of representatives for the isomorphism classes of simple representations of G. Show that

(a) r is the number of conjugacy classes in G.

(b) $(\dim V_1)^2 + \cdots + (\dim V_r)^2 = n$.

(7) If V and W are K-vector spaces with bases $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_n\}$ show that if we take $V \otimes W$ to be the K-vector space with basis $v_i \otimes w_j$ ($i = 1, \ldots, n$, $j = 1, \ldots m$) and define B by $B(v_i, w_j) = v_i \otimes w_j$, then B and $V \otimes W$ satisfy the coordinate-free defintion of tensor product.

(8) For $S \in \operatorname{End}_K V$, $T \in \operatorname{End}_K W$, show that $\operatorname{trace}(S \otimes T) = \operatorname{trace}(S) \operatorname{trace}(T)$.

(9) Suppose (ρ, V) is a representation of G and (σ, W) is a representation of H. Then $\rho \boxtimes \sigma : (g, h) \mapsto \rho(g) \otimes \sigma(h)$ is a representation of $G \times H$ on $V \otimes W$. Show that (without using characters and their properties) if ρ and σ are simple then so is $\rho \boxtimes \sigma$. Moreover, if ρ' and σ' are simple representations

of G and H respectively, such that ρ' is not isomorphic to ρ and σ' is not isomorphic to σ, then $\rho \boxtimes \sigma$ is not isomorphic to $\rho' \boxtimes \sigma'$.

(10) If $V' = \mathrm{Hom}_K(V, K)$ is the dual vector space of V then for any vector space W the linear map $V' \otimes W \to \mathrm{Hom}_K(V, W)$ induced by the bilinear map $V' \times W \to \mathrm{Hom}_K(V, W)$ defined by

$$(\xi, w) \mapsto (v \mapsto \xi(v)w)$$

is an isomorphism of vector spaces.

(11) Let $\beta : V' \otimes V \to \mathrm{End}_K V$ be the linear map of the previous exercise (in the case where $W = V$). Let τ be the linear map $V' \otimes V \to K$ induced by the bilinear map $(\xi, v) \mapsto \xi(v)$ (from $V' \times V$ to K). Recall that the trace map trace : $\mathrm{End}_k V \to K$ is defined as the sum of diagonal entries of the matrix corresponding to a linear map with respect to any basis. Show that $\mathrm{trace} \circ \beta = \tau$.

(12) Let (ρ, V) be an irreducible representation of G. Define $\Phi : V' \otimes V \to K[G]$ by

$$\Phi(\xi, v) = \sum_{x \in G} \xi(\rho(x)v) 1_x$$

Then Φ is an injective intertwiner of representations of $G \times G$, where $(g, h) \in G \times G$ acts on $\xi \otimes v \in V' \otimes V$ by $(\rho'(g)\xi) \otimes (\rho(g)v)$ and on $T \in \mathrm{End}_K V$ by $\rho(g)^{-1} \circ T \circ \rho(h)$.

(13) Assume that the characteristic of the lagebraically closed field K does not divide $|G|$. Using the explicit Wedderburn decomposition, i.e., the fact that the primitive central idempotents in $K[G]$ are given by

$$\epsilon_i(g) = \frac{\dim V_i}{|G|} \mathrm{trace}(\rho_i(g); V_i)$$

where $(\rho_1, V_1), \ldots, (\rho_r, V_r)$ are a set of representatives for the simple representations of G over K, prove the basic properties of the irreducible characters of G, namely, if $\chi_i(g) := \mathrm{trace}(\rho_i(g), V_i)$, then

(a) $\chi_i, \ldots, \chi_r$ form a basis for the centre of $K[G]$ (the class functions).

(b) $|G|^{-1} \sum_{g \in G} \chi_i(g)\chi_j(g^{-1}) = \delta_{ij}$ for all i, j (orthogonality relations).