# REPRESENTATION THEORY

ASSIGNMENT DUE ON AUGUST 22 2011

(1) Prove that Definition 1 and Definition 2 of the group algebra result in isomorphic algebras.
(2) If $R$ is a finite dimensional $K$-algebra, then every $R$-module has a finite dimensional invariant subspace (in particular, every simple $R$-module is finite dimensional).
(3) *Find a necessary and sufficient condition on a field $K$ and a finite group $G$ so that the subspace
$$K[G]_0 = \Big\{ f : G \to \mathbf{C} : \sum_{g \in G} f(g) = 0 \Big\}$$
has an invariant complement in the regular representation $(L, K[G])$.
(4) Let $H$ be a subgroup of $G$. View $K[G]_0$ as a representation of $H$ by restricting $L$ to $H$. List all the invariant complements of this representation of $H$ on $K[G]_0$.
(5) Let $n > 1$ be an integer. Show that $K[\mathbf{Z}/n\mathbf{Z}]$ is isomorphic to $K[t]/(t^n - 1)$.
(6) *Suppose that $K$ has characteristic $p$. Find all the invariant subspaces for the regular representation of $K[Z/p^k\mathbf{Z}]$.
(7) Let $V$ be a finite dimensional vector space over $K$. Then $V$ is tautologically an $\mathrm{End}_K V$ module (take $\tilde{\rho}$ to be the identity map $\mathrm{End}_K V \to \mathrm{End}_K V$). Show that $V$ is a simple $\mathrm{End}_K V$-module.
(8) The kernel and image of an intertwiner are invariant subspaces.
(9) Let $V$ be as above and fix $T \in \mathrm{End}_K V$. Then $V$ can be viewed as a $K[t]$-module (here $K[t]$ denotes the algebra of polynomials in the free variable $t$ with coefficients in $K$) via the $K$-algebra homomorphism $\tilde{\rho} : K[t] \to \mathrm{End}_K V$ determined by $t \mapsto T$. Show that $\mathrm{End}_{K[T]} V$ consists of linear maps $T \to T$ such that $ST = TS$.
(10) Suppose that $(\rho_1, V_1)$ and $(\rho_2, V_2)$ are simple representations over a field $K$ (which need not be algebraically closed). Let $T : V_1 \to V_2$ be an intertwiner. Show that, if $T$ is non-zero, then it is an isomorphism.
(11) *Let $T$ be a self-intertwiner of a simple representation (the underlying field need not be algebraically closed). Show that the minimal polynomial of $T$ is irreducible.
(12) If $R$ is a $K$-algebra such that every module admits a finite dimensional invariant subspace and an invariant complement (for example, $R$ could be a finite dimensional semisimple algebra), then every $R$-module (not necessarily finite dimensional) is a sum of simple modules (this uses Zorn's lemma).
(13) Show that every finite dimensional complex representation of a finite group admits an invariant Hermitian inner product (i.e., $\langle \rho(g)x, \rho(g)y \rangle = \langle x, y \rangle$ for all $x$, $y$ and $g$). Use this to prove Maschke's theorem for $K = \mathbf{C}$.