$x_1, x_2, \dots, x_n$ at random

$x_i \in U[0,1]$

$$P(x_i \in (\alpha, \beta)) = \beta - \alpha$$

$0 \le \alpha \le \beta \le 1$

x_1

When does $\vec{x} = (x_1, x_2, \dots, x_n)$ give the identity?

$$P(x_1 \le x_2 \le \dots \le x_n) = ?$$

$$\int_0^1 \int_{x_1}^1 \int_{x_2}^1 \dots \int_{x_{n-1}}^1 \boxed{dx_n \, dx_{n-1} \dots dx_2 \, dx_1}$$

$$= \frac{1}{n!}$$

$$= \int_{x_1 \le x_2 \le \dots}$$

$\{x_1 \le \dots \le x_n\} = \{x_1 \le x_2, x_2 \le x_3, \dots$

$$\left.\begin{array}{l} i < j \\ \omega(i) > \omega(j) \end{array}\right\} (i,j) \text{ is an inversion.}$$

Inversion table:

$$\omega = \omega(1)\,\omega(2) \dots \omega(n)$$

$$\alpha = \alpha_1 \;\; \alpha_2 \dots \alpha_n$$

$$\alpha_1 = \#\{\cancel{j<1}^{\,i<1} \mid \cancel{(1,j)}^{\,(i,1)} \text{ is an inversion}\}$$

$$\alpha_2 = \#\{i<2 \mid (i,2) \text{ is an inversion}\}$$

$$\alpha_3 = \#\{i<3 \mid (i,3) \text{ is an inversion}\}$$

$$\vdots$$

$$\alpha_n = \#\{i<n \mid (i,n) \text{ is an inversion}\}$$

$$\omega = 6\,3\,1\,7\,2\,5\,4$$
$$\alpha = 0\,1\,2\,0\,3\,2\,3 \xrightarrow{\Sigma} 11 \text{ inversions}$$

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{matrix}$$

$w' = \boxed{2571634}$

$\alpha' = 0003133$

$+ \quad \alpha = 0120323$

$\alpha'' = 0123456$

$w'' = 7654321$

$\alpha''' = 0000000$

$w''' = 1234567$

$\alpha \in \{0\} \times \{0,1\} \times \{0,1,2\} \times \cdots \{0,1,\ldots,n-1\}$

$\alpha = 0120323$

$\cancel{1}\cancel{2}\ 3\ 4\ \cancel{5}\ 6\ 7$

6317254

$\mathrm{inv}(w) = \#\{i<j \mid w(i) > w(j)\}$

= no. of inversions of w

$$c(n,k) = \#\{w \in S_n \mid \mathrm{inv}(w) = k\}$$

$$= \#\{\underbrace{(\alpha_1, \dots, \alpha_n)}_{\cap\ \mathbb{Z}^n} \mid 0 \le \alpha_j \le j-1,\ \sum \alpha_j = k\}$$

123	0
132	1
213	1
231	2
312	2
321	3

$$\boxed{\alpha_1\ \alpha_2\ \dots,\alpha_{n-1}}\ \boxed{\alpha_n}$$

$c(1,k)$

$c(1,0) = 1$

$c(2,0) = 1$, $c(2,1) = 1$

$c(3,0) = 1$, $c(3,1) = 2$, $c(3,2) = 2$

$c(3,3) = 1$

$$c(n,k) = \sum_{r=0}^{n-1} c(n-1, k-r)$$

$$\sum_{k=-\infty}^{+\infty} c(n,k)\, t^k = \sum \left(\sum c(n-1, \right.$$

$$\binom{n}{2} \quad k=0$$

$$= \sum_{k=-\infty}^{+\infty} \sum_{r=0}^{n-1} c(n-1, k-r)\, t^{k-r}\, t^r$$

$$= \sum_{r=0}^{n-1} t^r \sum_{k=-\infty}^{+\infty} c(n-1, k) t^k$$

$$F_n(t) = \sum_{k=-\infty}^{\infty} c(n,k) t^k = \sum_{0}^{\binom{n}{2}}$$

$$F_n(t) = (1+t+t^2+\cdots+t^{n-1}) F_{n-1}(t)$$

$$F_1(t) = 1$$

$$F_2(t) = (1+t)$$

$$F_3(t) = 1(1+t)\boxed{(1+t+t^2)}$$

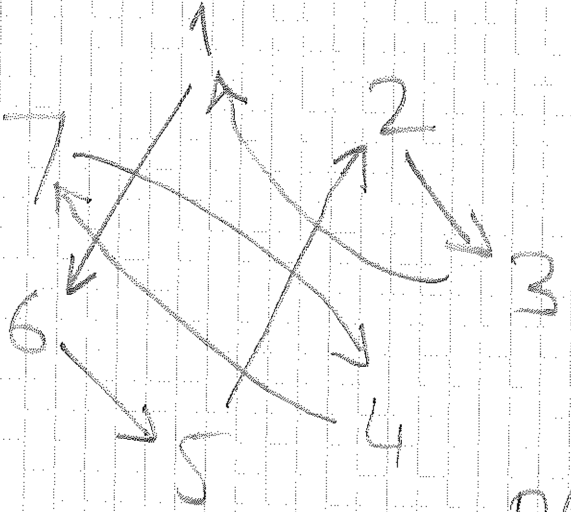
$$F_n(t) = \prod_{j=0}^{n-1} \frac{1-t^{j+1}}{1-t} = \boxed{\prod_{j=0}^{n-1} \left(\sum_{k=0}^{j} x^k \right)}$$

Symmetric unimodal

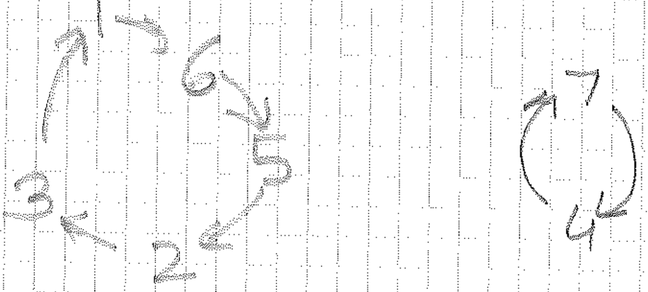
$a_0 + a_1 t + \cdots + a_N t^N$ degree N

- $a_i = a_{N-i}$
- $a_i \leq a_{i+1}$ if $i \leq \frac{N}{2}$

$$\begin{array}{ccccccccc} & & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ w & = & 6 & 3 & 1 & 7 & 2 & 5 & 4 \end{array}$$



$p(n)$



$\boxed{(6\,5\,2\,3\,1)(7\,4)}$ — cycle decomposition of w

$(6\,5\,2\,3\,1|7\,4$

- Start each cycle with largest el.
- arrange cycles in ~~decreasing~~ increasing order of largest element

$(6\,3\,1)(7\,2\,5\,4)$ $\quad 6\,5\,1\,7\,4\,3\,2$

__Cycle types__ - list of cycle sizes in weakly decreasing order

Integer partn of n $\quad \boxed{\lambda_1 \geq \ldots \geq \lambda_\ell \quad \lambda_1 + \ldots + \lambda_\ell = n}$