Relative Positions in Finite Abelian Groups

Amritanshu Prasad

Institute Seminar Week 2011

Spaces and Symmetry

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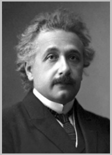
Galileo


Galois


Jordan


Lie


Klein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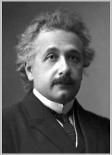
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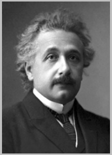

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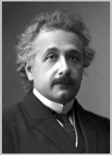
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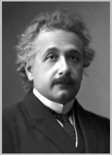
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The theory concerns quantities which are invariant under the symmetry group.

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Thus, the fundamental invariant of Euclidean geometry, namely distance, is a complete invariant of relative position.

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- Λ depends on A only through the combinatorial invariants of its structure (and not the specific primes dividing its order)

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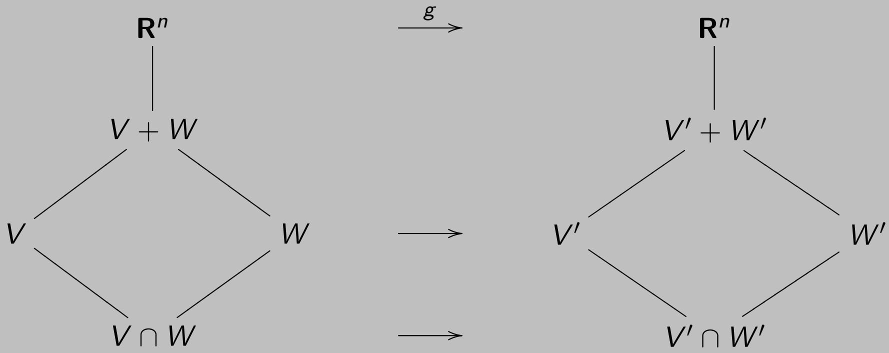
> *The pairs* (V, W) *and* (V', W') *have the same relative position if and only if* $\dim(V \cap V') = \dim(W \cap W')$.

This can be visualized as follows:



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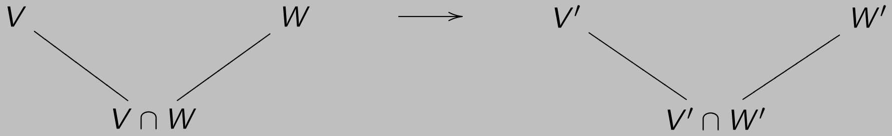

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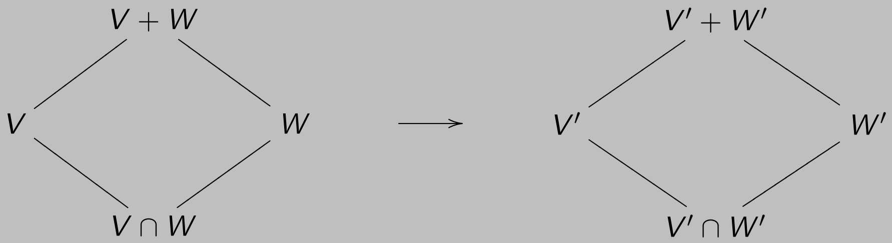

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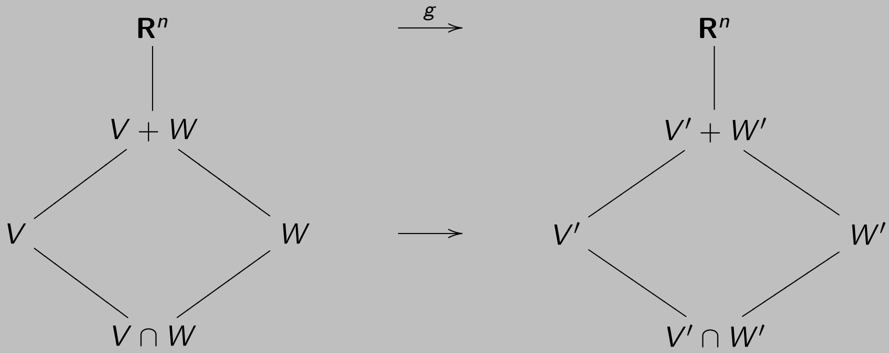

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In general, this problem is known to be a *wild classification problem*.

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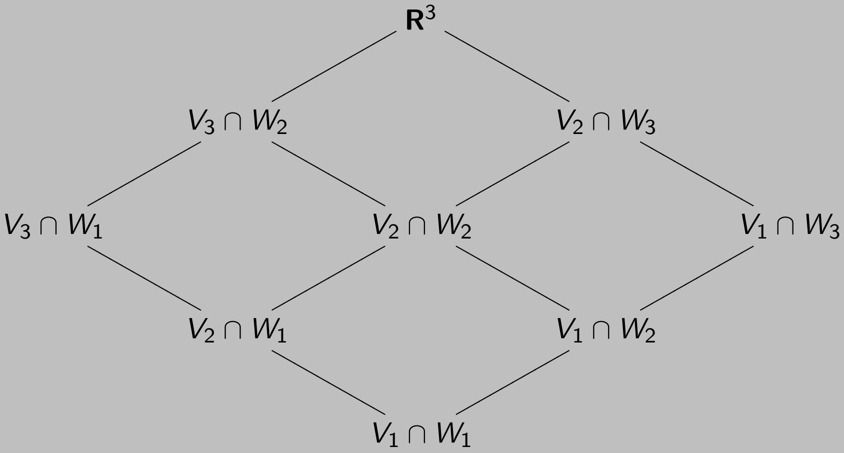

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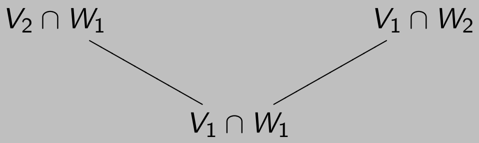

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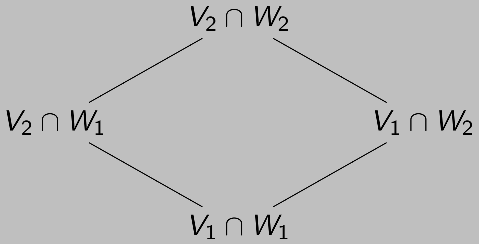

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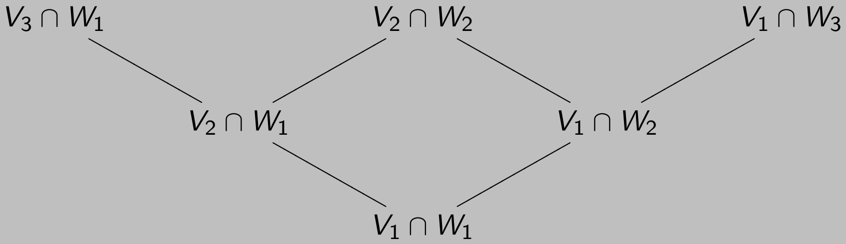

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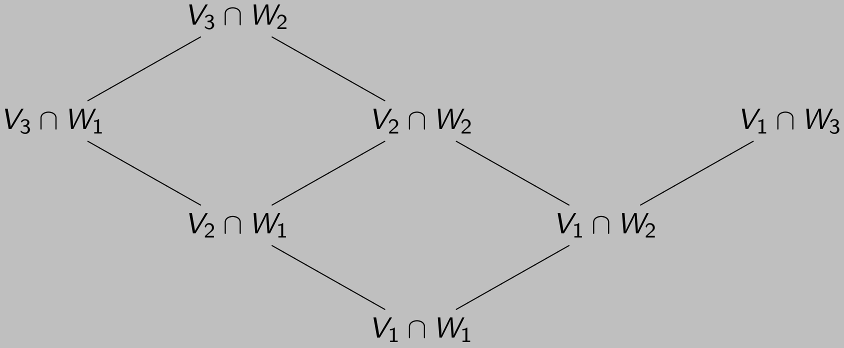

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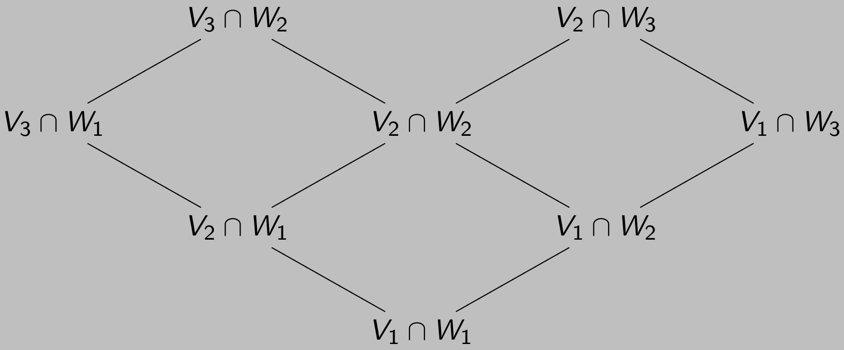

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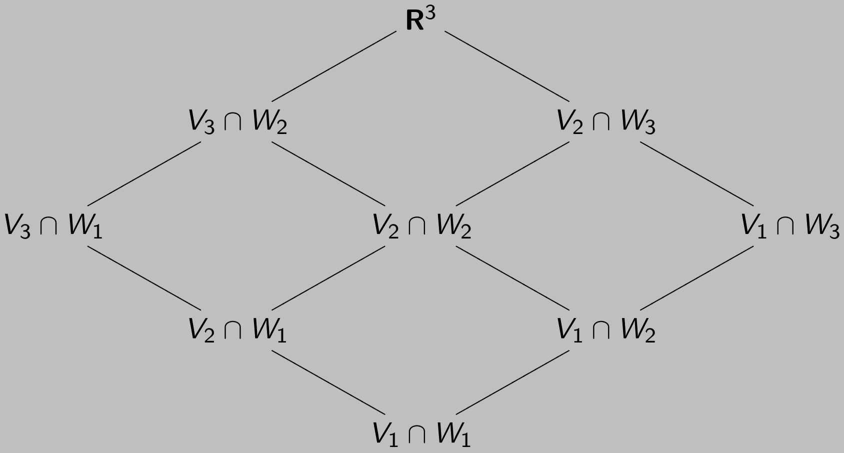

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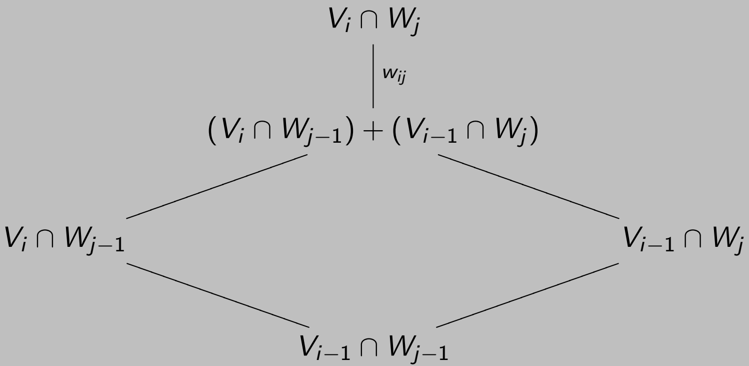
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Therefore, for each j, there exists unique i such that $w_{ij} = 1$ (for all other values of i, $w_{ij} = 0$.
Similarly, for each i, there is a unique j such that $w_{ij} = 1$.
Thus $w = (w_{ij})$ is a permutation matrix.

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The relative position of (V, W) is completely determined by w, where

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- Combinatorial identities involving Gaussian binomial coefficients

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- $G = GL(n, \mathbf{R})$, the group of invertible matrices

Beyond the Bruhat decomposition

Let $R = \mathbf{Z}/p^k\mathbf{Z}$.
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The strategy of systematically extending isomorphisms of subspaces fails because even if two finite abelian groups are isomorphic, not every isomorphism of subgroups extends to an isomorphism of the groups.

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This is called the matrix of **intersection numbers** (Onn, Prasad and Vaserstein, 2006)

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- Make the connections with the representation theory of $GL_n(R)$