q-rious positivities in orbit problems

Amritanshu Prasad

The Institute of Mathematical Sciences, Chennai

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Positivity in q-binomial coefficients

$$\binom{n}{k}_q = \text{no. of } k\text{-dimensional subspacess of } \mathbf{F}_q^n$$
$$= \frac{(q^n-1)(q^n-q)\cdots(q^n-q^{n-k+1})}{(q^k-1)(q^k-q)\cdots(q^k-q^{k-1})}.$$

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Recurrence relation

$$\binom{n+1}{k}_q = q^k\binom{n}{k}_q + \binom{n}{k-1}_q.$$

q-Meru Prastaara

1

$1 \quad 1$

$1 \quad q + 1 \quad 1$

$1 \quad q^2 + q + 1 \quad q^2 + q + 1 \quad 1$

$1 \quad q^3 + q^2 + q + 1 \quad q^4 + q^3 + 2q^2 + q + 1 \quad q^3 + q^2 + q + 1 \quad 1$

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Theorem (Knuth)

The coefficient of q^i in $\binom{n}{k}_q$ is the number of partitions of i with largest part at most $n-k$ and at most k parts.

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Example

$$\binom{7}{3}_q =$$

$$q^{12}+q^{11}+2q^{10}+3q^9+4q^8+4q^7+5q^6+4q^5+4q^4+3q^3+2q^2+q+1.$$

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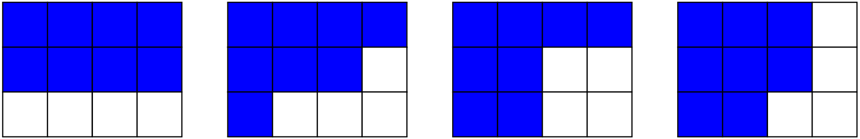
Coefficient of q^8

The number of partitions of 8 that fit in a $(7-3)\times 3$ box.



Orbit problem: counting similarity classes

Similarity

Matrices A and B are said to be *similar*, written $A \sim B$, if there exists an invertible matrix X such that

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Count the number of similarity classes in $M_n(\mathbf{F}_q)$.

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Enumerative problem

Count the number of similarity classes in $M_n(\mathbf{F}_q)$.

n	classes
2	$q^2 + q$
3	$q^3 + q^2 + q$
4	$q^4 + q^3 + 2q^2 + q$
5	$q^5 + q^4 + 2q^3 + 2q^2 + q$
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Rational canonical form

Let C_p denote the companion matrix of a monic polynomial p.

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$$p(t) = t^k + a_1 t^{k-1} + \cdots + a_{k-1} t + a_k,$$

then

$$C_p = \begin{pmatrix} 0 & 0 & \dots & -a_k \\ 1 & 0 & \dots & -a_{k-1} \\ 0 & 1 & & -a_{k-2} \\ & & \ddots & \\ 0 & 0 & \dots & -a_1 \end{pmatrix}$$

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Every matrix in $M_n(\mathbf{F}_q)$ is similar to a unique matrix of the form:

$$\begin{pmatrix} C_{p_1} & & & \\ & C_{p_2} & & \\ & & \ddots & \\ & & & C_{p_k} \end{pmatrix}$$

where $p_1, \dots, p_k$ are polynomials in $\mathbf{F}_q[t]$, with $p_1 | p_2 | \cdots p_k$ and $\deg p_1 + \cdots + \deg p_k = n$.

Counting classes

Similarity classes in $M_n(\mathbf{F}_q)$

$$\updownarrow$$

$p_1, \ldots, p_k$ in $\mathbf{F}_q[t]$ with $\sum_i \deg p_i = n$, $p_1 | p_2 | \cdots | p_k$

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Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition of n.

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Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition of n.
No. of $p_1 | \cdots | p_k$ with $\deg p_i = \lambda_{k-i+1}$ is:

$$q^{\lambda_k + (\lambda_{k-1} - \lambda_k) + \cdots + (\lambda_1 - \lambda_2)} = q^{\lambda_1}.$$

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Conclusion

The number of similarity classes in $M_n(\mathbf{F}_q)$ is

$$c_n = \sum_{\lambda \vdash n} q^{\lambda_1}.$$

n	c_n
2	$q^2 + q$
3	$q^3 + q^2 + q$
4	$q^4 + q^3 + 2q^2 + q$
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Theorem

The coefficient of q^k in c_n is the number of partitions of n with largest part equal to k.

Matrix Pairs

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Observation

The number $c_{n,2}$ is a monic polynomial in q with non-negative integer coefficients and degree $n^2 + 1$.

How did we compute $c_{n,2}$?

Burnside's lemma

$$|G\backslash X| = \frac{1}{|G|}\sum_{g\in G} |X^g|.$$

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$$\begin{aligned}
c_{n,2} &= |GL_n(\mathbf{F}_q)\backslash M_n(\mathbf{F}_q)\times M_n(\mathbf{F}_q)| \\
&= \frac{1}{|GL_n(\mathbf{F}_q)|}\sum_{g\in GL_n(\mathbf{F}_q)}|M_n(\mathbf{F}_q)^g|^2 \\
&= \frac{1}{|GL_n(\mathbf{F}_q)|}\sum_{g\in GL_n(\mathbf{F}_q)}|Z_{M_n(\mathbf{F}_q)}(g)|^2.
\end{aligned}$$

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The sum on the last line can be computed using the classification of conjugacy classes in $GL_n(\mathbf{F}_q)$ via Jordan canonical form.

Classes in $M_n(\mathbf{F}_q)$

$$\text{similarity classes in } GL_n(\mathbf{F}_q)$$

$$\updownarrow$$

$$\rho : \mathrm{Irr}\mathbf{F}_q[t] \to \Lambda, \ \sum_{p \in \mathrm{Irr}\mathbf{F}_q[t]} \deg p |\rho(p)| = n.$$

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The class corresponding to ρ is invertible if and only if $\rho(t) = \emptyset$.

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The class corresponding to ρ is invertible if and only if $\rho(t) = \emptyset$.
We have:

- $\dim Z_{M_n(\mathbf{F}_q)}(\rho) = \sum_p \sum_{i,j} \min(\rho(p)_i, \rho(p)_j)$,
- $|Z_{GL_n(\mathbf{F}_q)}(\rho)| = |Z_{M_n(\mathbf{F}_q)}(\rho)| \prod_p \prod_i \prod_{j=1}^{m_i(\rho(p))} (1 - q^- j)$.
- Number of irreducible polynomials of degree d in $\mathbf{F}_q[t]$ is

$$\Phi_d(q) = \frac{1}{d} \sum_{e|d} \mu(d/e) q^d.$$

Similarity Class Type

Given $\rho : \mathrm{Irr}\mathbf{F}_q[t] \to \Lambda$, $|Z_{M_n(\mathbf{F}_q)}(\rho)|$ and $|Z_{GL_n(\mathbf{F}_q)}(\rho)$ do not depend on the exact polynomials $p \in \mathrm{Irr}\mathbf{F}_q[t]$, but only on their degree.

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Type of ρ

The type of ρ is the multiset of symbols:

$$\tau = \{\rho(p)_{\deg p} \mid \rho(p) \neq \emptyset\}.$$

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Example: Types of degree 2

1. $\{(2)_1\}$ - non-semisimple matrices
2. $\{(1,1)_1\}$ - central matrices
3. $\{(1)_2\}$ - irreducible matrices
4. $\{(1)_1, (1)_1\}$ - non-central diagonalizable matrices

Type Invariants

Given a type

$$\tau\{\lambda_{d_1}^{(1)}, \cdots, \lambda_{d_k}^{(k)}\},$$

1. The centralizer algebra of a matrix of type τ in $M_n(\mathbf{F}_q)$ is:

$$Z_\tau = \prod_{i=1}^{k} q^{d_i \sum_{r,s} \min(\lambda_r^{(i)}, \lambda_s^{(i)})}.$$

2. The centralizer group of a matrix of type τ in $GL_n(\mathbf{F}_q)$ is:

$$z_\tau = Z_\tau \prod_{i=1}^{k} \prod_{j} \prod_{l=1}^{m_l(\lambda^{(i)})-1} (1 - q^{-l}).$$

3. The number of matrices of type τ is given by:

$$n_\tau = \prod_{d} \binom{\Phi_d(q)}{n_d(\tau)},$$

where $n_d(\tau)$ is the number of elements of τ with subscript d.

How did we compute $c_{n,2}$?

We get:

$$c_{n,2} = \frac{1}{|GL_n(\mathbf{F}_q)|} \sum_{\rho} \frac{|GL_n(\mathbf{F}_q)|}{|Z_{GL_n(\mathbf{F}_q)}(\rho)|} |Z_{GL_n(\mathbf{F}_q)}(\rho)|^2$$

$$= \sum_{\tau} n_\tau^* \frac{Z_\tau(q)^2}{z_\tau(q)},$$

which is clearly a polynomial in q.

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$$= \sum_{\tau} n^*_{\tau} \frac{Z_{\tau}(q)^2}{z_{\tau}(q)},$$

which is clearly a polynomial in q.

This has been implemented in Sage, and was used to obtain the table:

n	$c_{n,2}$
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q-rious positivity

It follows from a result of Hausel, Letellier and Rodriguez-Villegas (*Annals* 2013), on the proof of the *Kac conjecture* that the coefficients of $c_{n,2}$ are non-negative integers.

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Open Question

What do they count?

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What do they count?

In other words, is there a simpler explanation for positivity?

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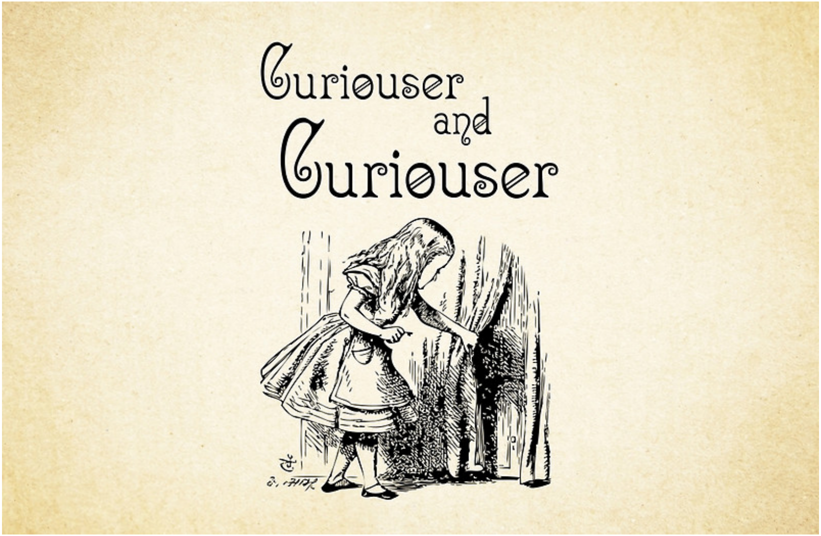
Open Question

What do they count?

In other words, is there a simpler explanation for positivity?
This question equally applies to $c_{n,k}$, the number of simultaneous simularity classes of k-tuples:

$$c_{n,k} = \sum_{\tau} n_{\tau}^{*} \frac{Z_{\tau}(q)^k}{z_{\tau}(q)}$$


Curiouser
and
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Commuting tuples

Let

$$M_n^{(k)}(\mathbf{F}_q) = \{(A_1, \ldots, A_k) \in M_n(\mathbf{F}_q)^k \mid A_i A_j = A_j A_i \text{ for all } i, j\}$$

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Let $g_{n,k}(q)$ be the number of simultaneous similarity classes in $M_n^{(k)}(\mathbf{F}_q)$.

$$\begin{aligned}
g_{1,2}(q) &= q^2 \\
g_{2,2}(q) &= q^4 + q^3 + q^2 \\
g_{3,2}(q) &= q^6 + q^5 + 2q^4 + q^3 + 2q^2 \\
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Uday has shown that $g_{n,k}(q)$ is a polynomial with non-negative integer coefficients for $n \leq 4$ and all k (by exact enumeration).

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\end{aligned}$$

Uday has shown that $g_{n,k}(q)$ is a polynomial with non-negative integer coefficients for $n \leq 4$ and all k (by exact enumeration). In general, it has not been proven that $g_{n,k}(q)$ is a polynomial in q.

Pairs in Abelian Groups

$$A_{p,\lambda} = \mathbf{Z}/p^{\lambda_1}\mathbf{Z} \oplus \cdots \mathbf{Z}/p^{\lambda_l}\mathbf{Z}.$$

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$$c_\lambda^{(k)}(p) = |G_{p,\lambda}\backslash A_{p,\lambda}^k|.$$

- $c_\lambda^{(1)}(p)$ is independent of p (with Kunal Dutta, JCTA 2011).
- $c_\lambda^{(2)}(p)$ is a polynomial in p with integer coefficients of degree λ_1.

n = 1	
(1)	$q + 2$
n = 2	
(2)	$q^2 + 2q + 2$
(1, 1)	$q + 3$
n = 3	
(3)	$q^3 + 2q^2 + 2q + 2$
(2, 1)	$q^2 + 5q + 5$
(1, 1, 1)	$q + 3$
n = 4	
(4)	$q^4 + 2q^3 + 2q^2 + 2q + 2$
(3, 1)	$q^3 + 5q^2 + 7q + 4$
(2, 2)	$q^2 + 3q + 5$
(2, 1, 1)	$q^2 + 5q + 6$
(1, 1, 1, 1)	$q + 3$
n = 5	
(5)	$q^5 + 2q^4 + 2q^3 + 2q^2 + 2q + 2$
(4, 1)	$q^4 + 5q^3 + 7q^2 + 6q + 4$
(3, 2)	$q^3 + 5q^2 + 10q + 7$
(3, 1, 1)	$q^3 + 5q^2 + 8q + 6$
(2, 2, 1)	$q^2 + 6q + 8$
(2, 1, 1, 1)	$q^2 + 5q + 6$
(1, 1, 1, 1, 1)	$q + 3$

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This conjecture has been verified by computer for all partitions of $n \leq 19$.

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Wide open

What about $c_\lambda^{(k)}(p)$ for $k > 2$? We do not even have a proof that it is a polynomial in p.