Polynomials as Characters of Symmetric Groups

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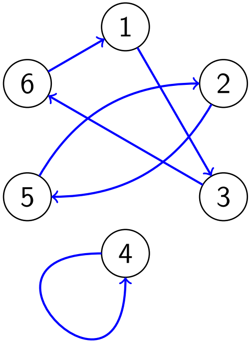

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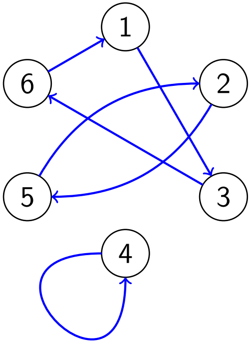

has cycle type $(3, 2, 1)$.

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$$n = |\alpha| = \sum_i i a_i.$$

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$$\text{Probability}(w \text{ has cycle type } \alpha) = \frac{1}{z_\alpha}.$$

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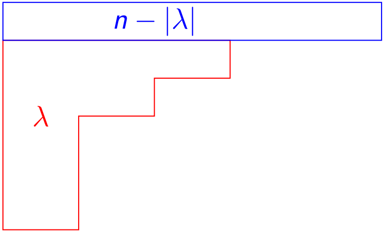

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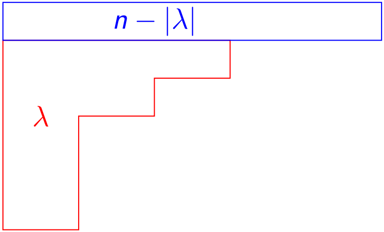

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Here $V_{\mu[n]}$ denotes the irreducible representations (Specht module) corresponding to $\mu[n]$.

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Remark

$\mathbf{E}_n f$ and $\mathbf{F}_n f$ are the expected values of the random variables $f(w)$ and $\operatorname{sgn}(w)f(w)$ respectively, when w is chosen uniformly at random from S_n.

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Here $\left\{\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right\} = \prod_{i\geq 1}\left\{\begin{smallmatrix}a_i\\ b_i\end{smallmatrix}\right\}$ is a product of Stirling numbers of the second kind, and $\beta = 1^{b_1}2^{b_2}\cdots$. Also, $\beta! = \prod_{i\geq 1} b_i!$.

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Theorem (Murnaghan's stability theorem)

The Kronecker coefficients $g^{\lambda[n]}_{\mu[n]\nu[n]}$ *are independent of* n *for* $n \gg 0$.

Turning the ~~Tables~~ Tableaux

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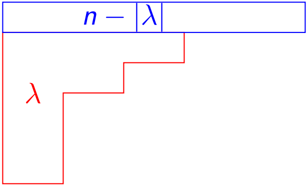
$\lambda[n] =$



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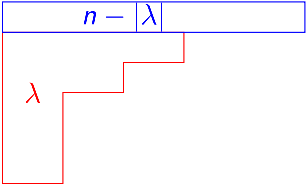

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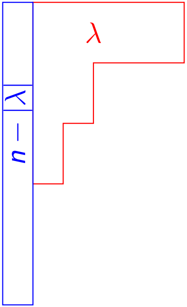

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For $n \gg 0$, this is 0.

Moment Generating Functions

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Let $\mathbf{N} := \{0, 1, 2, \ldots, \}$.

$$\sum_{n\in\mathbf{N}} \sum_{\lambda\in\mathbf{N}^l} \mathbf{F}_n H_\lambda t^\lambda v^n = \prod_{R\sqsubset[l]} (1 + t^R v),$$
$$\sum_{n\in\mathbf{N}} \sum_{\lambda\in\mathbf{N}^l} \mathbf{E}_n H_\lambda t^\lambda v^n = \prod_{R\sqsubset[l]} (1 - t^R v)^{-1},$$
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References

The main reference for this talk is:

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Also see:

Specht modules decompose as alternating sums of restrictions of schur modules, by Assaf and Speyer, Proc. Amer. Math. Soc., 2019. doi:10.1090/proc/14815.

For a nice discussion of Specht character polynomials, see:

A. M. Garsia and A. Goupil. Character polynomials, their q-analogs and the Kronecker product. Electron. J. Combin., 16(2, Special volume in honor of Anders Bjrner):Research Paper 19, 40, 2009. doi:10.1016/j.jcta.2019.02.019.