MACKEY'S LITTLE GROUP METHOD

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ABSTRACT. Wigner and Mackey's "little group method" for constructing representations is described for finite groups.

1. Induced Representations

Let G be a finite group and H any subgroup. Let $\sigma : H \to U(\mathcal{H}_\sigma)$ be a unitary representation. H acts on the right of the product $G \times \mathcal{H}_\sigma$ by

$$(g, x) \cdot h = (gh, \sigma(h^{-1})x) \text{ for all } h \in H,\ g \in G,\ x \in \mathcal{H}_\sigma.$$

The quotient space $\mathcal{E}_\sigma = (G \times \mathcal{H}_\sigma)/H$ is a vector bundle over G/H. Let $\mathcal{H}_\pi$ be the space of all sections over G/H of this bundle. G acts on $\mathcal{H}_\pi$ by

$$(\pi(g)s)(\gamma) = s(g^{-1}\gamma) \text{ for all } g \in G, \text{ and } \gamma \in G/H.$$

The resulting representation π of G on $\mathcal{H}_\pi$ is called *the representation of G induced from the representation σ of H*. We write $\pi = \operatorname{Ind}_H^G \sigma$.

Proposition 1.1. *Let $\operatorname{Ind}_H^G \mathcal{H}_\sigma$ be the space of all functions $\tilde{s} : G \to \mathcal{H}_\sigma$ satisfying*

$$\tilde{s}(gh) = \sigma(h^{-1})\tilde{s}(g).$$

Then the map $s(gH) = (g, \tilde{s}(g))H$ is a well-defined section of $\mathcal{E}_\sigma$. Moreover, when G is made to act on $\operatorname{Ind}_H^G \mathcal{H}_\sigma$ by

$$g' \cdot \tilde{s}(g) = \tilde{s}(g'^{-1}g),$$

The map $\tilde{s} \mapsto s$ is an isomorphism of G-representations.

Proof. To check that s is well-defined, suppose that $h \in H$. Then $(gh, \tilde{s}(gh)) = (gh, \sigma(h^{-1})\tilde{s}(g))$, which lies in the same H-orbit of $G \times \mathcal{H}_\sigma$ as $(g, \tilde{s}(g))$.

Given a section s of $\mathcal{E}_\sigma$, define $\tilde{s}$ so that $s(g) = (g, \tilde{s}(g))H$ for each g in G. This defines an inverse to the map $\tilde{s} \mapsto s$. □

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2. Systems of Imprimitivity

Let G be a finite group, and $\pi : G \to U(\mathcal{H}_\pi)$ a unitary representation.

Definition 2.1. A *system of imprimitivity* for π is a G-set Γ together with a decomposition

$$\mathcal{H}_\pi = \bigoplus_{\gamma\in\Gamma} \mathcal{H}_\gamma$$

such that $\pi(g)\mathcal{H}_\gamma \subseteq \mathcal{H}_{g\gamma}$.

Example 2.2. If N is a normal subgroup of G, then G acts on the set $\hat{N}$ of equivalence classes of unitary irreducible representations of N by $(g\gamma)(n) = \gamma(g^{-1}ng)$ (the representation space of $g\gamma$ is also taken to be $\mathcal{H}_\gamma$; clearly, the action is trivial when restricted to N). Let

$$\mathcal{H}_\gamma = \{\pi(\epsilon_\gamma)x | x \in \mathcal{H}_\pi\},$$

where $\epsilon_\gamma \in \mathbf{C}[N]$ is the central idempotent corresponding to $\gamma \in \hat{N}$. Then

$$\mathcal{H}_\pi = \bigoplus_{\gamma\in\hat{N}} \mathcal{H}_\gamma.$$

For any $s \in G$ and $x \in \mathcal{H}_\pi$, we have

$$\begin{aligned} \pi(g)(\pi(\epsilon_\gamma)x) &= \pi(g)\pi(\epsilon_\gamma)\pi(g^{-1})\pi(g)x \\ &= \pi(\epsilon_{g\gamma})\pi(g)x, \end{aligned}$$

and so $\pi(g)\mathcal{H}_\gamma \subset \mathcal{H}_{g\gamma}$.

Example 2.3. Let H be any subgroup of G, and let $\sigma : H \to U(\mathcal{H}_\sigma)$ be a unitary representation. Let $\pi = \operatorname{Ind}_H^G \sigma$ and let $\mathcal{H}_\gamma$ be the fibre of $\mathcal{E}_\sigma$ (see §1) over $\gamma \in G/H$. Then

$$\mathcal{H}_\pi = \bigoplus_{\gamma\in G/H} \mathcal{H}_\gamma$$

is a system of imprimitivity for π.

Definition 2.4. The system of imprimitivity $\mathcal{H}_\pi = \bigoplus_{\gamma\in\Gamma} \mathcal{H}_\gamma$ is *transitive* if the action of G on Γ is transitive.

Example 2.3 is an example of a transitive system of imprimitivity.

3. Mackey's Imprimitivity Theorem

Theorem 3.1 (Mackey's Imprimitivity Theorem)**.** *Let π be any representation of G, and suppose $\mathcal{H}_\pi = \bigoplus_{\gamma\in\Gamma} \mathcal{H}_\gamma$ is a transitive system of imprimitivity for π. For any point $\gamma \in \Gamma$, let G_γ denote the stabiliser of γ in G. Let π_γ denote the representation of G_γ on $\mathcal{H}_\gamma$. Then π is isomorphic to $\operatorname{Ind}_{G_\gamma}^G \pi_\gamma$.*

Proof. The map $gG_\gamma \mapsto g\gamma$ induces an isomorphism $G/G_\gamma \to \Gamma$ of G-sets. Therefore, any $x \in \mathcal{H}_\pi$ can be written uniquely as

$$x = \sum_{gG_\gamma \in G/G_\gamma} x_{g\gamma}, \text{ with } x_{g\gamma} \in H_{g\gamma} \text{ for each coset } gG_\gamma.$$

Moreover, $g^{-1}x_{g\gamma} \in \mathcal{H}_\gamma$. Therefore, one may define a map $\Phi : \mathcal{H}_\pi \to \operatorname{Ind}_{G_\gamma}^G \mathcal{H}_\gamma$ by $(\Phi(x))(g) = \pi(g^{-1})x_{g\gamma}$. It is not too difficult to verify that this is a homomorphism of G-representations $\pi \to \operatorname{Ind}_{G_\gamma}^G \pi_\gamma$. On the other hand, given a section s of $\mathcal{E}_{\pi_\gamma}$ over G/G_γ, define $\Psi(s) \in \mathcal{H}_\pi$ by

$$\Psi(s) = \sum_{gG_\gamma \in G/G_\gamma} \pi(g)s(gG_\gamma).$$

This, once again is a homomorphism of G-representations, and Φ and Ψ are mutual inverses. □

4. Application to representation theory

Suppose that G is a finite group and that N is a normal subgroup of G. Given any finite dimensional representation π of G write

$$\mathcal{H}_\pi = \bigoplus_{\gamma \in \hat{N}} \mathcal{H}_\gamma,$$

as in Example 2.2. Let

$$\Gamma = \{\gamma \in \hat{N} | \mathcal{H}_\gamma \neq 0\}.$$

It follows from Example 2.2 that Γ is G-stable.

Lemma 4.1. *If π is irreducible, then Γ is a transitive G-set.*

Proof. Take any $\gamma \in \Gamma$. Let

$$U' = \bigoplus_{g \in G} \mathcal{H}_{g\gamma}.$$

Then U' is a non-trivial G-stable subspace of $\mathcal{H}_\pi$. Since π is irreducible, $U' = \mathcal{H}_\pi$. □

We have proved the following

Proposition 4.2. *Let π be an irreducible representation of G. Let Γ and $\mathcal{H}_\gamma$ be as above. Then*

$$\mathcal{H}_\pi = \bigoplus_{\gamma \in \Gamma} \mathcal{H}_\gamma$$

is a transitive system of imprimitivity for π. Consequently, by Mackey's imprimitivity theorem, $\pi \cong \operatorname{Ind}_{G_\gamma}^G \pi_\gamma$, where π_γ is the representation of G_γ on $\mathcal{H}_\gamma$ obtained by restricting π to G_γ.

Lemma 4.3. *Let G be a group, N a normal subgroup of G, and $\gamma \in \hat{N}$. If $\pi_\gamma \in \hat{G}_\gamma$ is such that its restriction to N is γ-isotypic, then $\operatorname{Ind}_{G_\gamma}^G \pi_\gamma$ is irreducible.*

Proof. By Frobenius reciprocity,

$$(*) \qquad \operatorname{End}_G(\operatorname{Ind}_{G_\gamma}^G \pi_\gamma) = \operatorname{Hom}_{G_\gamma}(\pi_\gamma, \operatorname{Res}_{G_\gamma}^G \operatorname{Ind}_{G_\gamma}^G \pi_\gamma).$$

On the other hand,

$$\operatorname{Res}_{G_\gamma}^G \operatorname{Ind}_{G_\gamma}^G \pi_\gamma \cong \bigoplus_{G_\gamma x G_\gamma \in G_\gamma \backslash G / G_\gamma} \pi_\gamma^x.$$

Now, π_γ^x, when restricted to N is γ^x isotypic. Therefore π_γ^x is not isomorphic to π_γ unless $x \in G_\gamma$. It follows that the spaces in $(*)$ are one dimensional, and hence that $\operatorname{Ind}_{G_\gamma}^G \pi_\gamma$ is irreducible. □

Combining Proposition 4.2 with Lemma 4.3 gives

Theorem 4.4. *The function $\pi \mapsto \Gamma$ is a surjective map from $\hat{G}$ to $\hat{N}/G$. The fibre over the orbit of $\gamma \in \hat{N}$ is in bijective correspondence with the irreducible representations of G_γ whose restriction to N is γ-isotypic.*

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