Counting and Symmetry

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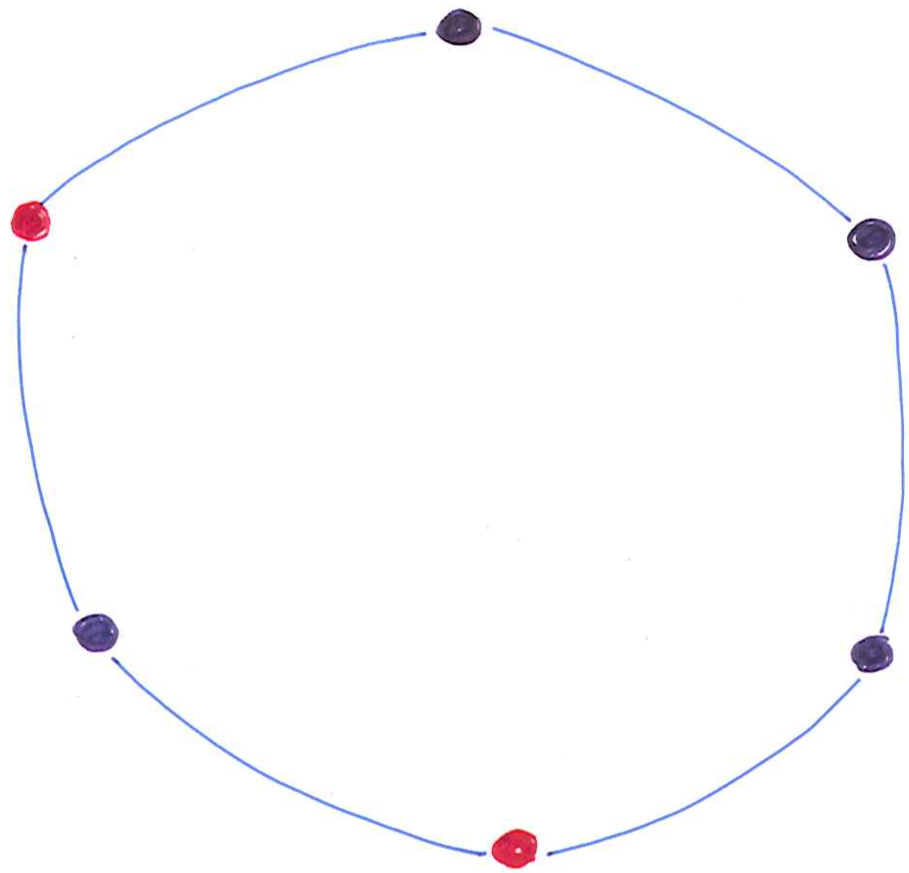
Reference: Culture, Excitement, and relevance of Mathematics, by V. Krishnamurthy
Wiley Eastern Ltd., 1990.

Tamil Translation by T. Geetha



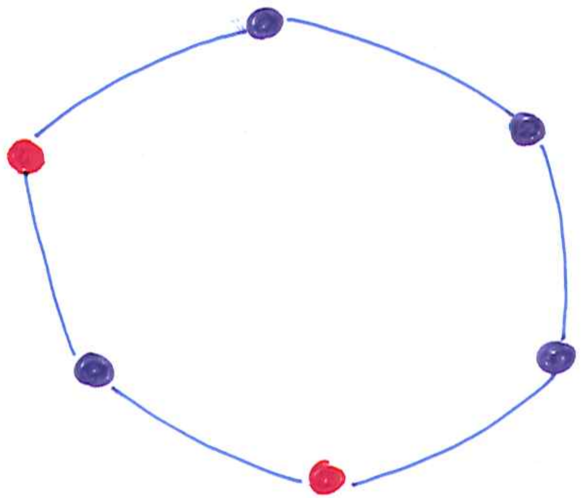
A necklace with 4 blue, and 2 red beads

நான்கு நீலம் மற்றும் இரண்டு சிவப்பு மணிகள்

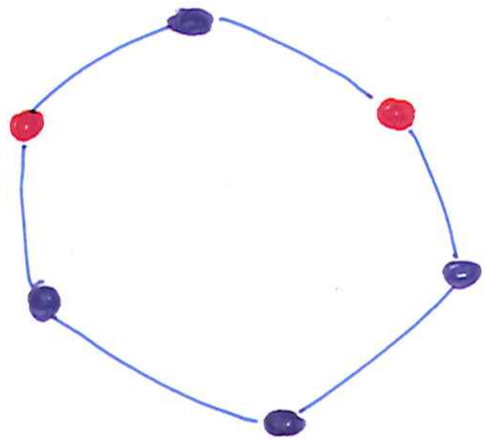
உள்ள நெக்லெஸ்

இரண்டு நிறங்களிலுள்ள ஆறு மணிகளைக் கொண்ட நெக்லெஸ் எத்தனை உள்ளது?

How many necklaces with 2 colours and 6 beads?



and



(1) (2)

are really the same necklace

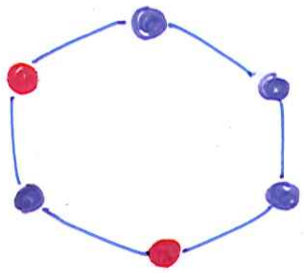
(rotation moves one to the other)

இரண்டுமே ஒரே நெக்லெஸைக் குறிக்கிறது

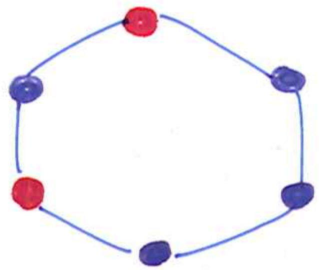
(சுழற்சி (1)ஐ (2)ற்கு எடுத்துச் செல்கிறது)

Effect of Rotation

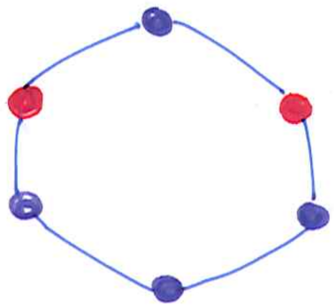
R_0



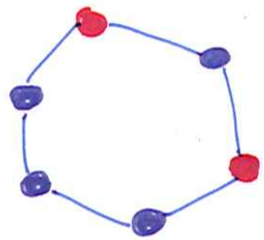
R_1



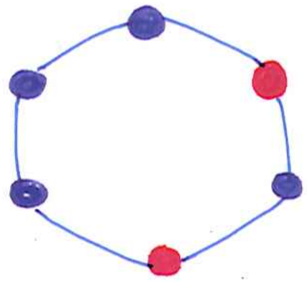
R_2



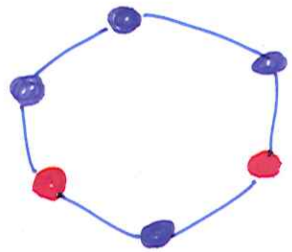
R_3



R_4

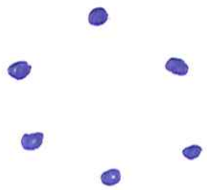


R_5

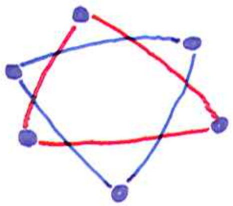


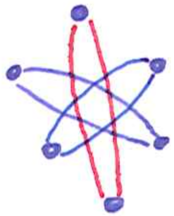
சுழற்சியின் விளைவு

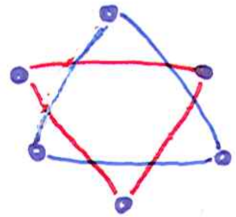
Orbits

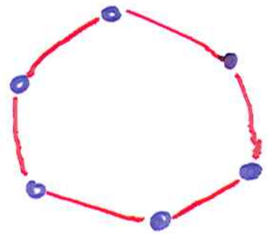
R_0  6

R_1  1

R_2  2

R_3  3

R_4  2

R_5  1

சுற்றுப்பாதைகள்

Invariance – சுழற்சியின் போது
மாறாதது

$$\boxed{\text{Number of invariances} = 2^{\text{number of orbits}}}$$

Rotation	no. of orbits	no. of invariances
R_0	6	64
R_1	1	2
R_2	2	4
R_3	3	8
R_4	2	4
R_5	1	2
		$\underline{\underline{84}}$

Number of distinct necklaces

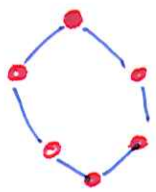
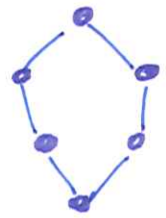
$$= \frac{1}{6} \sum_{i=0}^{5} (\text{no. of invariances of } R_i)$$

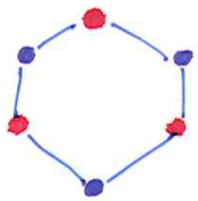
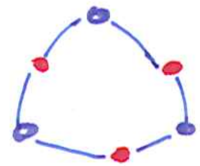
$$= \frac{84}{6} = 14$$

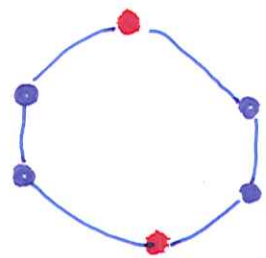
Burnside's theorem: each necklace is counted 6 times as an invariance

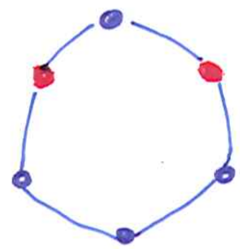
பர்ன்சைடின் தேற்றம் : ஒவ்வொரு நெக்லெஸ்சும் invarianceன் 6 மடங்காக எண்ணப்படுகிறது.

Period of a necklace = no. of distinct rotations

 and  have period 1

 has period = 2  

What is the period of  ?

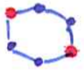
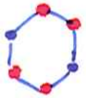
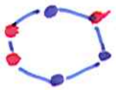
 ?

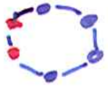
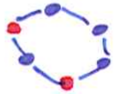
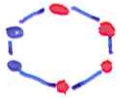
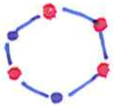
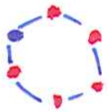
Period $\longleftrightarrow$ invariance of what?

Period	Invariance of	
1	$R_0, R_1, R_2, R_3, R_4, R_5$	$\times 1$
2	R_0, R_2, R_4	$\times 2$
3	R_0, R_3	$\times 3$
6	R_0	$\times 6$

Period ஐ அறிந்தால் எந்த சுழற்சிக்கு அது invariant என்பதை அறியலாம்.

Distinct 6 beaded necklaces with two colours

Pattern	Period	No. of times it appear as invariance of R_0	R_1	R_2	R_3	R_4	R_5	Total
	1	1	1	1	1	1	1	6
	1	1	1	1	1	1	1	6
	2	2	0	2	0	2	0	6
	3	3	0	0	3	0	0	6
	3	3	0	0	3	0	0	6
	6	6	0	0	0	0	0	6
	6	6	0	0	0	0	0	6

Pattern	Period	No. of times it appears as an invariance of R_0	R_1	R_2	R_3	R_4	R_5	Total
	6	6	0	0	0	0	0	6
	6	6	0	0	0	0	0	6
	6	6	0	0	0	0	0	6
	6	6	0	0	0	0	0	6
	6	6	0	0	0	0	0	6
	6	6	0	0	0	0	0	6
	6	6	0	0	0	0	0	6
		64	2	4	8	4	2	84

வேறுபட்ட அனைத்து நெக்லெஸ்கள்.

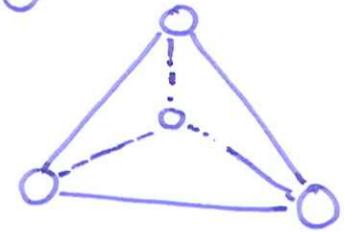
How many necklaces with 3 colours and 8 beads?

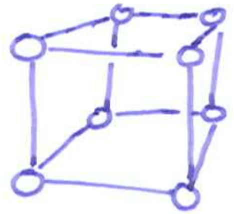
Rotation	No of orbits	No. of invariances
R_0	8	$3^8 = 6561$
R_1	1	3
R_2	2	9
R_3	1	3
R_4	4	81
R_5	1	3
R_6	2	9
R_7	1	3
		6672

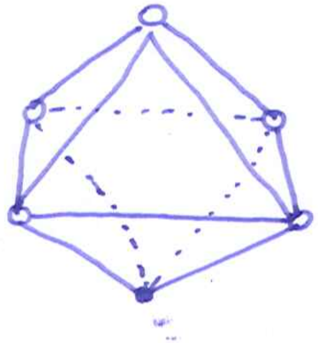
No. of necklaces $= \frac{6672}{8} = 834$!

Necklace problem with n beads

$\longrightarrow$ n symmetries (variant - 2n)

The tetrahedron :  - 24 symmetries

The cube :  - 48 symmetries

The octahedron :  - 48 symmetries

Symmetries form a group:

Group = Set + rule for composition

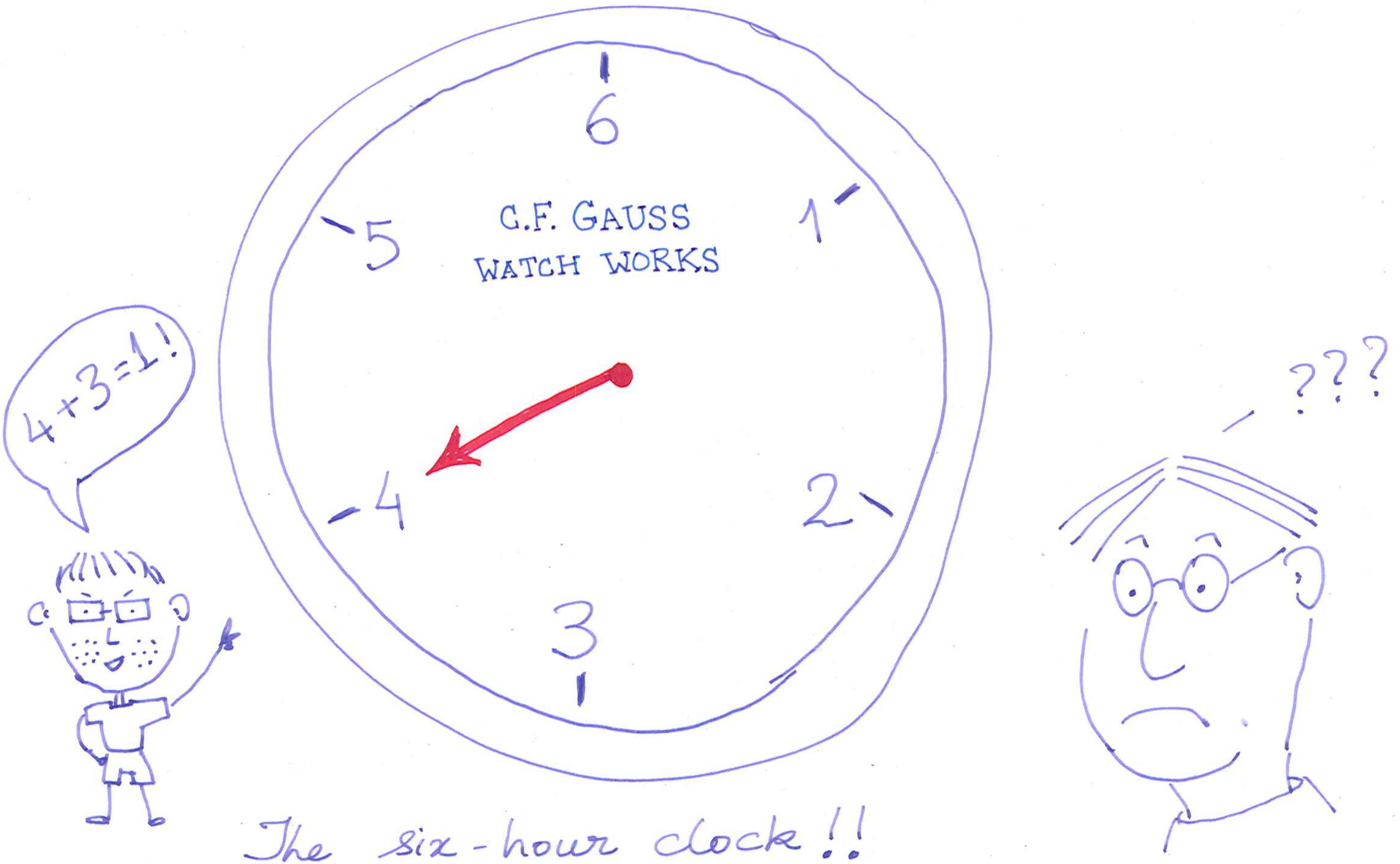
In the case of necklace with six beads:

$R_1 \circ R_1 = R_2$

$R_1 \circ R_1 \circ R_1 = R_3$

$R_4 \circ R_3 = ??$

Get the six-hour clock rule for arithmetic.



The six-hour clock!!

Group axioms: [Galois ; 1846] [Jordan] [Cayley]

A group is a set G, together with a function $\circ : G \times G \to G$ which satisfies:

1) Associativity : $(x \circ y) \circ z = x \circ (y \circ z)$ for all $x, y, z \in G$
2) Identity : There exists $e \in G$ such that $x \circ e = e \circ x$ for all $x \in G$
3) Inverse : For each $x \in G$, there exists $x^{-1} \in G$ such that $x \circ x^{-1} = e$

Two groups G_1 & G_2 are the "same" (isomorphic) if there exists a bijection $\varphi : G_1 \to G_2$ such that $\varphi(x \circ y) = \varphi(x) \circ \varphi(y)$.

Example: Symmetries of the cube & octahedron.

X : some finite set $\{1, 2, 3, 4, 5, 6\}$

G : some group of symmetries. $\{R_0, R_1, R_2, R_3, R_4, R_5\}$

C : set of colours $\{\text{red}, \text{blue}\}$

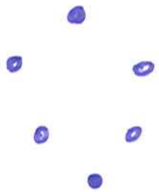
A colouring is a function: $f : X \to C$

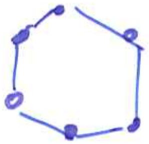
Two colourings f_1 & f_2 are equivalent if, for some $g \in G$

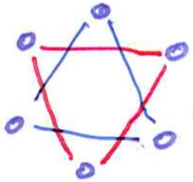
$$f_2(x) = f_1(g(x)) \text{ for all } x \in X$$

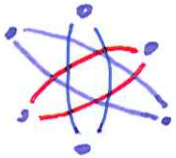
$\boxed{\text{Enumerate the number of equivalence classes of colourings.}}$

Cycle index of G :

R_0 :  $= 6$ 1-cycles $\longrightarrow$ s_1^6

R_5, R_1 :  $= 1$: 6-cycle $\longrightarrow$ $2s_6^1$

R_4, R_2 :  $= 2$ 3-cycles $\longrightarrow$ $2s_3^2$

R_3 :  $= 3$ 2-cycles $\longrightarrow$ s_2^3

Cycle index of G : $\frac{1}{6}\left[s_1^6 + 2s_6^1 + 2s_3^2 + s_2^3 \right]$

Introduce one symbol for each colour

two colour case: symbols: b & r

Substitute $s_1 \longrightarrow b + r$

$s_2 \longrightarrow b^2 + r^2$

$s_3 \longrightarrow b^3 + r^3$

etc.

into the cycle index: $\frac{1}{6}\left[s_1^6 + 2s_6^1 + 2s_3^2 + s_2^3\right]$

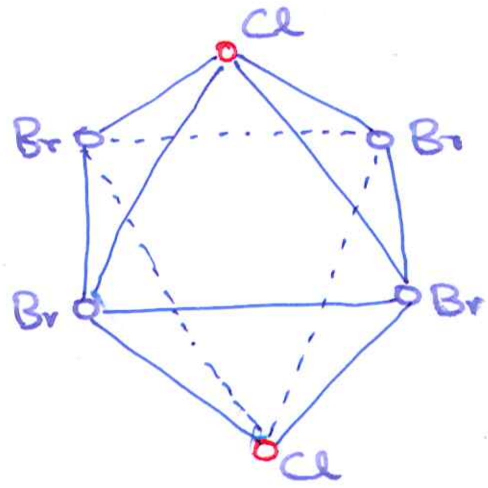
$\frac{1}{6}\left[(b+r)^6 + 2(b^6 + r^6) + 2(b^3 + r^3)^2 + (b^2 + r^2)^3\right]$

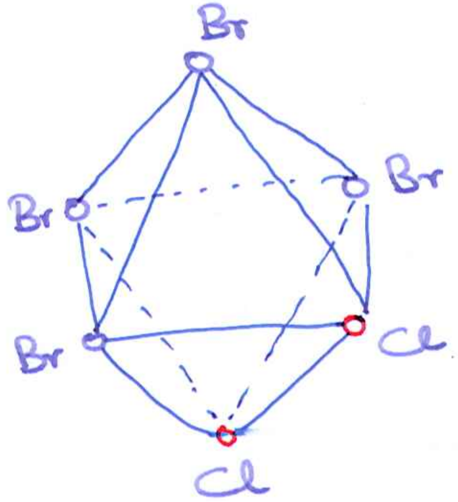
$= b^6 + b^5 r + 3b^4 r^2 + 4b^3 r^3 + 3b^2 r^4 + b r^5 + r^6$

Polya: coefficient of $b^i r^j$ = # of distinct necklaces with i blue beads and j red beads.

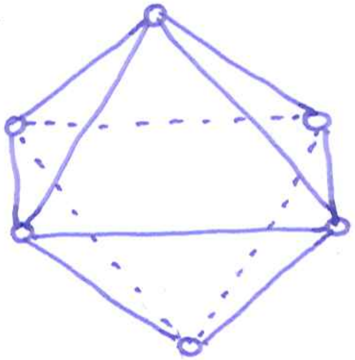
Application to chemistry: Two molecules are said to be isomers if they have the same atoms, but can not be transformed, one to another, by an ~~atomic~~ rotational symmetry.

Example: Octahedral isomers of $PtBr_4Cl_2$





Rotational symmetries of the octahedron :



Identity operation $\rightsquigarrow$ s_1^6

Six rotations about 3 axes
of symmetry joining pairs of $\rightsquigarrow 6s_1^2 s_4$
opposite vertices through 90°

Three rotations about the $\rightsquigarrow 3s_1^2 s_2^2$
same axes by 180°

Six rotations about six axes
of symmetry joining midpoints $\rightsquigarrow 6s_2^3$
of opposite edges, through 180°

Eight rotations about four axes
of symmetry joining midpoints $\rightsquigarrow 8s_3^2$
of opposite faces through 120°

Cycle index : $\frac{1}{24}\left[s_1^6 + 6s_1^2 s_4 + 3s_1^2 s_2^2 + 6s_2^3 + 8s_3^2\right]$

So, the no. of isomers of $Pt\,Br_4\,Cl_2$ is the coeff. of $r^4 b^2$ when we substitute $s_i = r^i + b^i$ in $\frac{1}{24}\left[s_1^6 + 6 s_1^2 s_4 + 3 s_1^2 s_2^2 + 6 s_2^3 + 8 s_3^2\right]$

$$\text{Ans} = \frac{1}{24}\left[\frac{6!}{4!\,2!} + 6 + 3\times 3 + 6\times 3 + 8\times 0\right]$$

$$= \frac{1}{24}\left[15 + 6 + 9 + 18\right]$$

$$= 2$$

Exercise: Calculate the no of octahedral isomers for $Co(NH_3)_2Cl_2Br_2$

$Co(NH_3)_4Cl_2$

$Co\,Cl_4\,\dot{H}\,(OH)$

$Co\,Cl\,Br\,(NO_2)(SCN)(H_2O)(NH_3)$