- Permutation reps & intertwiners
- Partition reps. & intertwiners
- Example : S_3
- RSK correspondence
- Classification of simple reps.
- How to go further.

G gp., X : G-set, K : field

$(\rho_X, K[X])$ - permutation rep.

$K[X] = \{\text{functions } X \to K\}$

$\rho_X(g) f(x) = f(g^{-1} \cdot x)$

ρ_X is a homom. $G \to GL(K[X])$

<u>Example</u>: $\lambda \vdash n$ (λ partition of n)

i.e., $\lambda = (\lambda_1, \dots, \lambda_\ell)$, $\lambda_1 \geq \dots \geq \lambda_\ell$, $\lambda_1 + \dots + \lambda_\ell = n$.

$$X_\lambda = \{(S_1, \dots, S_\ell) \mid \underset{\{1,\dots,n\}}{\underline{n}} = S_1 \sqcup \dots \sqcup S_\ell,\ |S_i| = \lambda_i\}$$

$G = S_n$. X_λ is an S_n-set.

$K[X_\lambda]$ is a representation of S_n.

②

(ρ, V), (σ, W) : representations of G

$T : V \longrightarrow W$ is called an <u>intertwiner</u> if

$T(\rho(g)v) = \sigma(g)T(v) \ \forall v \in V, g \in G.$

Notation: $T \in \mathrm{Hom}_G(V, W)$ [Category of reps.

Let's calculate: $\mathrm{Hom}_G(K[Y], K[X])$
X, Y : G-sets.

$$\mathrm{Hom}_k(K[Y], K[X]) = \{T_k \mid k \in K[X \times Y]\}$$

$$T_k f(x) = \sum_{y \in Y} k(x,y) f(y)$$ Integral operator with kernel k

$T_k \in \mathrm{Hom}_G(K[Y], K[X])$ iff ~~$\rho_X(g)\, k(x,y)\, \rho_Y(g)$~~

$$\rho_X(g)^{-1} \circ T_k \circ \rho_Y(g) = T_k$$

Now: $\rho_X(g)^{-1} \circ T_k \circ \rho_Y(g) f(x)$

$$= T_k \circ \rho_Y(g) f(gx)$$

$$= \sum_{y \in Y} k(gx, y) \cancel{f(y)}\, \cancel{\rho_Y(g) f(y)}\ f(g^{-1}y)$$

$$= \sum k(gx, gy) f(y)$$

$$= T_{k^g} f(x).$$

where $k^g(x,y) = k(gx, gy)$.

Conclusion: $T_k \in \mathrm{Hom}_G(K[Y], K[X])$ iff

$$k : G \backslash X \times Y \longrightarrow K.$$