Faces of the Fourier Transform

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Jean Baptiste Joseph Fourier (1768–1830)

"The deep study of nature is the most fruitful source of mathematical discoveries"

Life of Fourier

Born in Auxerre (Burgundy, France) in 1768
Ninth child of a tailor
Orphaned at age ten.
Trained for priesthood,
participated in the French revolution
was repeatedly arrested, risked the guillotine
taught at the École Polytechnique
accompanied Napoleon to Egypt
occupied important administrative and academic positions in France
helped write the *Description of Egypt*

Fourier's Magnum Opus

Théorie analytique de la chaleur
(The analytical theory of heat)
1822

"Heat, like gravity, penetrates every substance of the universe, its rays occupy all parts of space. The object of our work is to set forth the mathematical laws which this element obeys."

The heat equation

Temperature $v = v(x, t)$ as a function of space and time is governed by

$$\frac{\partial v}{\partial t} = \text{const.} \frac{\partial^2 v}{\partial x^2}$$

The Fourier transform

The Fourier transform of $f(x)$ is

$$\hat{f}(\xi) = \int f(x) e^{-2\pi i x \xi} dx$$

A function can be recovered from its Fourier transform

$$\hat{\hat{f}}(x) = f(-x)$$

Formal properties

Derivatives

If $g(x) = f'(x)$, then

$$\hat{g}(\xi) = -2\pi i \xi \hat{f}(\xi)$$

Products

If $g(x) = f_1(x) f_2(x)$, then

$$\hat{g}(\xi) = (\hat{f}_1 * \hat{f}_2)(\xi)$$

Application to the heat equation

The transformed heat equation

$$\partial \hat{v}/\partial t = -K\xi^2 \hat{v}(\xi)$$

has general solution

$$\hat{v}(\xi, t) = A(\xi)e^{-K\xi^2 t}$$

The inversion formula can be used to recover

$$v(x, t) = v_0 * G_t(x)$$

where $G_{Kt}(x) = Ce^{-Dx^2/t}$, and $v_0(x) = v(x, 0)$.

Fourier series

For a nice function $f : [0,1] \to \mathbf{C}$

$$f(x) = \sum_{n \in \mathbf{Z}} c_n e^{2\pi i n x},$$

where, for each n,

$$c_n = \int_0^1 f(x) e^{-2\pi i n x} dx.$$

Fourier assumed their validity for a "function" f.

Johann Peter Gustav Lejeune Dirichlet (1805-1859)

The modern definition of a function is due to Dirichlet.
He gave the first rigorous proof of the convergence of Fourier series, subject to what are now called *Dirichlet's conditions*.

Fourier transforms and Fourier series

Fourier transform

Associates to a function on $\mathbf{R}$ another function on $\mathbf{R}$.

Fourier series

Associates to a function on $\mathbf{R}/\mathbf{Z}$ a function on $\mathbf{Z}$, and vice versa.

Lev Semenovich Pontryagin (1908-1988)

Lost his eyesight to a stove explosion at age 14.
His mother read mathematical books and papers to him.
He developed duality theory for abelian groups.

Duality theory for abelian groups

Let A be a topological abelian group.
Its Pontryagin dual is the abelian group $\hat{A}$ consisting of continuous characters

$$\chi : A \to T$$

given the topology of uniform convergence on compact sets. Here

$$T = \{z \in \mathbf{C}^* : |z| = 1\}.$$

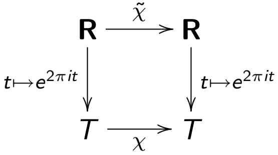
Dual of $\mathbf{Z}$

$\chi : \mathbf{Z} \to T$ is completely determined by the image of $1 \in \mathbf{Z}$.
Therefore,

$$\hat{\mathbf{Z}} = T.$$

Dual of T

Given $\chi : T \to T$, it lifts to unique $\tilde{\chi} : \mathbf{R} \to \mathbf{R}$:



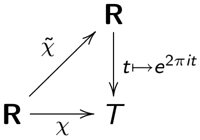
for which $\tilde{\chi}(0) = 0$.
If $\tilde{\chi}(1) = n$, then $n \in \mathbf{Z}$ and

$$\chi(z) = z^n \text{ for all } z \in T.$$

Therefore $\hat{T} = \mathbf{Z}$.

Dual of $\mathbf{R}$

Given $\chi : \mathbf{R} \to T$, it lifts to $\tilde{\chi} : \mathbf{R} \to \mathbf{R}$:



If $\tilde{\chi}(1) = \xi$, then $\chi = \chi_\xi$, where

$$\chi_\xi(t) = e^{2\pi i t \xi}$$

The map $\xi \mapsto \chi_\xi$ is an isomorphism of $\mathbf{R}$ onto $\hat{\mathbf{R}}$.

Dual of a finite group

A character $\mathbf{Z}/n\mathbf{Z} \to T$ is determined by the image of a generator, which can be any nth root of unity. Therefore $\mathbf{Z}/n\mathbf{Z}$ is its own Pontryagin dual.

Every finite abelian group is a product of finite cyclic groups.

Also, the dual of the product of two groups is the product of their duals.

Therefore every finite abelian group is isomorphic to its dual.

The double dual

There is always a map $\phi : A \to \hat{\hat{A}}$:

$$x \mapsto (\chi \mapsto \chi(x)).$$

Theorem (Pontryagin duality theorem)

When A is a locally compact abelian group, ϕ is an isomorphism of A onto $\hat{\hat{A}}$.

Examples of dual pairs

A	$\hat{A}$
$\mathbf{R}$	$\mathbf{R}$
$\mathbf{Z}$	T
$\mathbf{Z}/n\mathbf{Z}$	$\mathbf{Z}/n\mathbf{Z}$
$\mathbf{Z}(p^\infty)$	$\mathbf{Z}_p$
$\mathbf{Q}_p$	$\mathbf{Q}_p$
finite abelian group	finite abelian group
discrete abelian group	compact abelian group
compact connected	torsion-free discrete
compact totally disconnected	torsion discrete

The Fourier transform

Let A be a locally compact abelian group.
For $f \in L^1(A)$, its Fourier transform is is a function on $\hat{A}$:

$$\hat{f}(\chi) = \int_A f(x)\overline{\chi(x)}dx.$$

Fourier theory for locally compact abelian groups

1. For every locally compact abelian group A, the Fourier transform gives rise to an isometry of $L^2(A)$ onto $L^2(\hat{A})$.
2. If $\hat{\hat{A}}$ is identified with A using Pontryagin duality, the Fourier inversion formula holds:

$$\hat{\hat{f}}(x) = f(-x).$$

The discrete Fourier transform

For $f : \mathbf{Z}/n\mathbf{Z} \to \mathbf{C}$,

$$\hat{f}(k) = \frac{1}{n} \sum_{l=0}^{n-1} f(l) W^{lk}; \quad W = e^{2\pi i/n}.$$

The calculation of each value requires $n-1$ additions and $n+1$ multiplications,
so the calculation of $\hat{f}$ requires $2n^2$ operations.

Cooley-Tukey algorithm (1965)

Suppose $n = n_1 n_2$.
Write $k = n_1 k_1 + k_0$, $l = n_2 l_1 + l_0$.

$$\begin{aligned}
\hat{f}(n_1 k_1 + k_0) &= \sum_{l_0=0}^{n_2-1} \sum_{l_1=0}^{n_1-1} f(n_2 l_1 + l_0) W^{(n_1 k_1 + k_0)(n_2 l_1 + l_0)} \\
&= \sum_{l_0=0}^{n_2-1} f_1(l_0, k_0) W^{n_1 k_1 l_0}
\end{aligned}$$

(the inner sum over l_1 depends only on l_0 and k_0, since $W^{(n_1 k_1 + k + 0) n_2 l_1} = W^{k_0 n_2 l_1}$).

$$f_1(l_0, k_0) = \sum_{l_1=0}^{n_1-1} f(n_2 l_1 + l_0) W^{n_2 l_1 k_0 + k_0 l_0}$$

The inner array has n elements, each requiring $2nn_1$ operations.
To calculate the full sum requires an additional $2nn_2$ operations.
A total of $2n(n_1 + n_2)$ operations.

Cooley-Tukey repeated

If $n = 2^r$ then the above idea can be used recursively to calculate the Fourier transform with just

$$2n(2 + \cdots + 2) = 4nr = 4n \log_2 n$$

operations. Algorithms of this type are called fast Fourier transforms.

Polynomial multiplication

To multiply polynomials $a_0 + a_1x + \cdots + a_nx^n$ and $b_0 + b_1x + \cdots + b_mx^m$ naively requires $(m+1)(n+1)$ multiplication and $(m+1)(n+1) - (m+n+1)$ addition operations. Take $N \geq m+n$ and let $f, g : \mathbf{Z}/N\mathbf{Z} \to \mathbf{C}$ be given by

$$f(k) = a_k, \quad g(l) = b_l$$

(taking 0 for $k > n$ and $l > m$). The coefficient of x^j in the product is

$$\sum_{k+l=j} a_k b_l = f * g(j) = \widehat{\hat{f}\hat{g}}(-j)$$

which, using fast Fourier transform, requires only $12N \log_2 N + N$ operations.

The Poisson summation formula

For $f \in \mathcal{S}(\mathbf{R})$ (Schwartz class),

$$\sum_{n \in \mathbf{Z}} f(n) = \sum_{n \in \mathbf{Z}} \hat{f}(n).$$

For example, take $f_t(x) = e^{-\pi t x^2}$.
Then $\hat{f}_t(x) = \frac{1}{\sqrt{t}} f_{1/t}(x)$.
If we define

$$\Theta(t) = \sum_{n \in \mathbf{Z}} f_t(n),$$

then the Poisson summation formula gives

$$\Theta(t) = \frac{1}{\sqrt{t}} \Theta(1/t).$$

The Gamma function

For $\mathrm{Re}(s) > 0$,

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$$

defines a holomorphic function.
For $\mathrm{Re}(s) > 0$ the identity $s\Gamma(s) = \Gamma(s+1)$ holds.
This identity allows us to extend the Gamma function to a meromorphic function on on $\mathbf{C}$, with with simple poles at $0, -1, -2, \ldots$.

The Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \mathrm{Re}(s) > 1.$$

Analytic continuation and functional equation

The Riemann zeta function function extends to a meromorphic function on **C**, with a simple pole at $s = 1$ (and no other singularity), and $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ satisfies

$$\xi(s) = \xi(1 - s).$$

The proof

$$\begin{aligned}
\xi(s) &= \sum_{n=1}^{\infty} \int_0^{\infty} \pi^{-s/2} t^{s/2} e^{-t} n^{-s} \frac{dt}{t} \\
&= \sum_{n=1}^{\infty} \int_0^{\infty} \left(\frac{t}{n^2\pi}\right)^{s/2} e^{-t} \frac{dt}{t} \\
&= \sum_{n=1}^{\infty} \int_0^{\infty} t^{s/2} e^{-n^2\pi t} \frac{dt}{t} \quad [t \mapsto \frac{t}{n^2\pi}] \\
&= \int_0^{\infty} t^{s/2} \frac{1}{2}(\Theta(t) - 1) \frac{dt}{t}
\end{aligned}$$

Write

$$\int_0^\infty t^{s/2}\frac{1}{2}(\Theta(t)-1)\frac{dt}{t} = \int_0^1 t^{s/2}\frac{1}{2}(\Theta(t)-1)\frac{dt}{t}+\int_1^\infty t^{s/2}\frac{1}{2}(\Theta(t)-1)\frac{dt}{t}$$

Since $t \mapsto \frac{1}{2}(\Theta(t)-1)$ decreases rapidly as $t \to \infty$, the second integral is an entire function of s.

Substituting $t \mapsto 1/t$,

$$\begin{aligned}\int_0^1 t^{s/2}\frac{1}{2}(\Theta(t)-1)\frac{dt}{t} &= \int_1^\infty t^{-s/2}\frac{1}{2}\left(\Theta\left(\frac{1}{t}\right)-1\right)\frac{dt}{t} \\ &= \int_1^\infty t^{-s/2}\frac{1}{2}(\sqrt{t}\Theta(t)-1)\frac{dt}{t} \\ &= \int_1^\infty t^{-s/2}\frac{1}{2}(\sqrt{t}(\Theta(t)-1)+\sqrt{t}-1)\frac{dt}{t},\end{aligned}$$

which equals the sum of the entire function

$$\int_1^\infty t^{(1-s)/2}\frac{1}{2}(\Theta(t)-1)\frac{dt}{t}$$

and

$$\frac{1}{2}\int_1^\infty t^{(1-s)/2}\frac{dt}{t} - \frac{1}{2}\int_1^\infty t^{-s/2}\frac{dt}{t} = \frac{1}{s-1}-\frac{1}{s}.$$

We have

$$\xi(s) = \int_1^\infty (t^{\frac{s}{2}} + t^{\frac{1-s}{2}}) \frac{1}{2}(\Theta(t) - 1) \frac{dt}{t} - \frac{1}{s} - \frac{1}{1-s}.$$

Using the functional equation and knowing the poles of the Gamma function, we see that the Riemann zeta function has zeroes at even negative integers.
These are the so-called *trivial zeroes*.

Riemann hypothesis

All other zeroes should lie in the set $\mathrm{Re}(s) = \frac{1}{2}$.

"If I were to awaken after having slept for a thousand years, my first question would be: Has the Riemann hypothesis been proven?"

David Hilbert

Sources

1. *The analytical theory of heat*, by Joseph Fourier, Cambridge University Press.
2. *Fourier analysis*, by Thomas William Körner, Cambridge University Press.
3. *A first course in harmonic analysis*, by Anton Deitmar, Springer.

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