HOMEWORK VI

FUNCTIONAL ANALYSIS

(1) Show that there does not exist any function $f \in L^1_{\text{loc}}(\mathbf{R})$ such that for every $\phi \in \mathfrak{D}(\mathbf{R})$,
$$\int_{\mathbf{R}} f(x)\phi(x)dx = \phi(0).$$

(2) Let Ω be an open domain in $\mathbf{R}^n$, f_1 and f_2 be locally integrable functions on Ω. Show that $T_{f_1} = T_{f_2}$ if and only if $f_1 = f_2$ almost everywhere.

(3) For an arbitrary multi-index s, what are the distributional higher derivatives $D^s\delta_0$?

(4) Find all the distributions T on $\mathbf{R}$ for which the distributional derivative $\frac{d}{dx}T = 0$ [Hint: Firstly, note that a compactly supported smooth function $\phi(x)$ is the derivative of a compactly supported smooth function if and only if $\int \phi(x)dx = 0$. Fix $\phi_1(x) \in C_0^\infty(\mathbf{R})$ such that $\int \phi_1(x)dx = 1$. If $\phi(x) \in C_0^\infty(\mathbf{R})$, then we may write $\phi(x) = \phi_1(x)\int \phi(x)dx + \phi_0(x)$, for some $\phi_0 \in C_0^\infty(\mathbf{R})$ such that $\int \phi_0(x)dx = 0$.]

(5) Let $f(x) = f(x_1, \ldots, x_n)$ be a continuously differentiable function on a closed bounded domain Ω with a smooth boundary S. Define f to be zero outside Ω. Prove Green's theorem:
$$(\Delta T_f)(\phi) = T_{\Delta f}(\phi) + \int_S \frac{\partial f}{\partial \nu}\phi(x)ds - \int_S f(x)\frac{\partial \phi}{\partial \nu}ds,$$
where $\Delta = \sum_j \partial/\partial x_j$ and $\partial/\partial\nu = \sum_j (\partial/\partial x_j)\cos(x_j, \nu)$. Here (x_j, ν) denotes the angle between the inner normal to S and the jth coordinate vector.

Date: due on Monday, February 18, 2008 (before class).