HOMEWORK IV

FUNCTIONAL ANALYSIS

(1) Five an example of a sequence of functions $\{f_n\}$ from $\mathbf{R} \to [0,1]$ such that
  (a) each f_n is continuous on $\mathbf{R}$,
  (b) $f_n(x)$ converges to $f(x)$ for all x, $\int |f_n(x) - f(x)|^p dx \to 0$ for every $p \in (0, \infty)$, and
  (c) f is discontinuous at some point in $\mathbf{R}$.

(2) Find the conditions under which equality holds in Hölder's inequality.

(3) Find the conditions under which equality holds in Minkowski's inequality.

(4) Given $(S, \mathfrak{B}, m)$ a set, a σ-field and a measure, call a measurable function $x : S \to K$ *essentially bounded* if there exists a constant α such that $|x(s)| \leq \alpha$ almost everywhere in m. The infimum of all such constants is called the *essential supremum* of x, denoted $\text{ess sup}_{s\in S} |x(s)|$ or $\|x\|_\infty$. If $m(S) < \infty$ show that

$$\lim_{p\to\infty} \left(\int_S |x(s)|^p m(ds) \right)^{1/p} = \text{ess sup}_{s\in S} |x(s)|.$$

(5) Give an example to show that the result in the previous problem fails if $m(S) = \infty$.

(6) Give an example of a bounded real-valued function f on $\mathbf{R}$ such that there is no sequence of continuous functions $\{f_n\}$ such that $\|f_n - f\|_\infty \to 0$.

(7) Given $(S, \mathfrak{B}, m)$ as above, suppose that f is a measurable real-valued function on S such that $\int_S f dm$ exists. Show that $\phi(B) = \int_B f dm$ defines a signed measure on $(S, \mathfrak{B})$.

(8) Given f as in the previous problem, let $f^+ = \max\{f, 0\}$, $f^- = \min\{f, 0\}$. Show that if $B \in \mathfrak{B}$,

$$\overline{V}(\phi; B) = \int_B f^+ dm, \quad \underline{V}(\phi; B) = \int_B f^- dm, \quad V(\phi; B) = \int_B |f| dm.$$

(9) A signed measure $\phi \in A(S, \mathfrak{B})$ is said to be *concentrated* on a set $B \in \mathfrak{B}$ if $\phi(B') = 0$ for every $B' \in \mathfrak{B}$ such that $B \cap B' = \emptyset$. Show that if ϕ is concentrated on B then so is its total variation.

(10) Suppose $\phi \in A(S, \mathfrak{B})$ is a signed measure and $\phi = \phi_1 - \phi_2$, where ϕ_1 and ϕ_2 are [non-negative] measures. Show that $\phi_1(B) \geq \overline{V}(\phi; B)$ and $\phi_2(B) \geq -\underline{V}(\phi; B)$ for every $B \in \mathfrak{B}$.

Date: due on Monday, February 4, 2008 (before class).