HOMEWORK X

FUNCTIONAL ANALYSIS

(1) Show that $f \in \mathcal{S}(\mathbf{R}^2)$ is invariant under all transformations of the form $(x, y) \mapsto (ax + by, cx + dy)$ with $ad - bc = 1$ if and only if $\hat{f}$ is. As a corollary, show that $P(D)\delta_0$ is invariant under all transformations of variables of the same form if and only if $P(\xi)$ is a constant polynomial.
(2) Give an example of a compact operator which is not Hilbert-Schmidt.
(3) If E is a Hilbert space and $T : E \to E$ is a compact operator. Show that $T(E)$ does not contain a closed infinite dimensional subspace.
(4) Suppose X is a finite measure space and K_1, $K_2 \in L^2(X \times X)$. Let T_{K_i} be the integral operator corresponding to K_i, $i = 1, 2$. Show that $T_{K_1} \circ T_{K_2}$ is also an integral operator.
(5) Suppose E is a Hilbert space and $T : E \to E$ is compact. Show that there exists a sequence $\{T_n\}$ of finite rank operators such that $\|T - T_n\| \to 0$.
(6) Let T and S be bounded operators on a Banach space. Show that if one of T and S is compact, then so is $T \circ S$.
(7) Suppose T is a bounded operator on a Hilbert space. Show that T is compact if and only if T^* is.
(8) Show that the space of all Hilbert-Schmidt operators on a Hilbert space is again a Hilbert space.
(9) Show that

$$e^{-(x-\lambda)^2} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} (-1)^n \frac{d^n e^{-x^2}}{dx^n}.$$

Date: due on Monday, March 31, 2008 (before class).