The Frobenius Characteristic of Character Polynomials

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This article is a quick introduction to the theory of character polynomials which allow us study characters if symmetric groups of S_n across all n. The main conceptual tool is the formula in § 2.5 for the total Frobenius characteristic of a character polynomial. Formulas for the moments of character polynomials were first obtained in [3]. Their variants for signed moments are new.

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1. Preliminaries

In this section we recall basic facts about symmetric functions. A quick expositions of these facts can be found in Macdonald [2] or Prasad [4, 5]. As usual, $\mathbf{Q}$ denotes the field of rational numbers. Following Anglo-French combinatorialists' notation, we use $\mathbf{N}$ to denote the set of non-negative integers and $\mathbf{P}$ to denote the set of positive integers.

1.1. Integer Partitions. An integer partition is a finite weakly decreasing sequence of positive integers

$$\lambda = (\lambda_1, \dots, \lambda_l), \quad \lambda_1 \geq \dots \geq \lambda_l.$$

The size of the integer partition λ is the sum $|\lambda|$ of its parts $\lambda_1, \dots, \lambda_l$. Note that there is one integer partition of size 0, namely the empty partition with no parts, denoted $\emptyset$.

It is often convenient to express the integer partition λ in *exponential notation*

$$\lambda = 1^{l_1} 2^{l_2} \cdots ,$$

where l_i is the number of parts of λ that are equal to i. We have

$$|\lambda| = \sum_{i\in\mathbf{P}} il_i.$$

Integers which do not occur in the partition may be omitted from the exponential notation. For example, the partition $(4,4,2,1,1,1)$ has exponential notation $1^3 2^1 4^2$. We denote the set of all integer partitions by Par and the set of integer partitions of size n by Par_n.

Given an integer partition $\lambda = (\lambda_1, \dots, \lambda_l)$ its Young diagram is the array $Y(\lambda) = \{(i,j) \mid 1 \leq i \leq l,\ 1 \leq j \leq \lambda_i\}$. The conjugate of a partition λ is the partition λ' whose Young diagram is $\{(j,i) \mid (i,j) \in Y(\lambda)\}$.

1.2. Conjugacy classes in S_n. Recall that S_n denotes the set of all permutations of the set $[n] = \{1, \dots, n\}$. An cycle of a permutation $w \in S_n$ is a minimal closed subset $C \subset [n]$ such that $w(C) = C$. The cycle C is said to be an i-cycle if it has cardinality i. The cycle type of a permutation $w \in S_n$ is the integer partition

$$\alpha = 1^{a_1} 2^{a_2} \cdots$$

where a_i is the number of i-cycles in w.

If w has cycle type α, then [5, Exercise 2.2.24] the centralizer of w in S_n has cardinality

$$z_\alpha := \prod_{i\in\mathbf{P}} i^{a_i} a_i!.$$

All the permutations of a given cycle type form a conjugacy class in S_n. By the orbit-stabilizer theorem, the number of permutations with cycle type α is $n!/z_\alpha$.

1.3. The Schur Inner Product. For each $n \in \mathbf{N}$, and class functions χ and η in S_n, consider the Schur inner product

$$\langle \chi, \eta \rangle_n = \frac{1}{n!} \sum_{w\in S_n} \chi(w)\eta(w).$$

With respect to this inner product, the irreducible characters of S_n form an orthonormal basis. By § 1.2,

$$\langle \chi, \eta \rangle_n = \sum_{\alpha\in\mathrm{Par}_n} \frac{1}{z_\alpha} \chi(w_\alpha)\eta(w_\alpha),$$

where for each $\alpha \in \mathrm{Par}_n$, w_α denotes a permutation with cycle type α.

1.4. Symmetric Functions. Let Λ denote the ring of symmetric functions in infinitely many variables $x_1, x_2, \dots$ and coefficients in $\mathbf{Q}$ [4, Section 1]. Then

$$\Lambda = \bigoplus_{d \in \mathbf{N}} \Lambda^d,$$

where Λ^d is the space of homogeneous symmetric functions of degree d. Here we are considering formal symmetric functions in infinitely many variables.

The algebra Λ is the polynomial algebra generated by the powersum symmetric functions

$$p_n = x_1^n + x_2^n + \cdots \text{ for } n \in \mathbf{P}$$

or the elementary symmetric functions

$$e_n = \sum_{i_1 < i_2 \cdots < i_n} x_{i_1} \cdots x_{i_n} \text{ for } n \in \mathbf{P},$$

or the complete symmetric functions

$$h_n = \sum_{i_1 \leq i_2 \leq < i_n} x_{i_1} \cdots x_{i_n} \text{ for } n \in \mathbf{P}.$$

It is natural to set $e_0 = h_0 = 1$.

For a partition $\lambda = 1^{l_1} 2^{l_2} \cdots$, powersum, elementary, and complete symmetric functions are defined by

$$p_\lambda = \prod_i p_i^{l_1}, \quad e_\lambda = \prod_i e_i^{l_i}, \quad h_\lambda = h_i^{l_i}.$$

The empty products $p_\emptyset = e_\emptyset = h_\emptyset = 1$.

The degree of p_λ, e_λ and h_λ is $|\lambda|$. The assertion that λ is the polynomial algebra generated by the elementary symmetric functions $\{p_i\}_{i \in \mathbf{P}}$, $\{e_i\}_{i \in \mathbf{P}}$, $\{h_i\}_{i \in \mathbf{P}}$ is equivalent to the assertion that

$$\{p_\lambda \mid \lambda \in \mathrm{Par}_d\}, \quad \{e_\lambda \mid |\lambda \in \mathrm{Par}_d\}, \quad \{h_\lambda \mid \lambda \in \mathrm{Par}_d\}$$

are bases of Λ^d for every $d \in \mathbf{N}$.

Let $\Lambda[[t]]$ denote the ring of formal power series with coefficients in Λ. We have generating functions

$$E(t) = \sum_{d \in \mathbf{N}} e_d t^d = \prod_{i \in \mathbf{P}} (1 + x_i t) \tag{1.4.1}$$

$$H(t) = \sum_{d \in \mathbf{N}} h_d t^d = \prod_{i \in \mathbf{P}} (1 - x_i t)^{-1} \tag{1.4.2}$$

$$P(t) = \sum_{d \in \mathbf{P}} p_d t^{d-1} = \sum_{i \in \mathbf{P}} \frac{x_i}{1 - x_i t}. \tag{1.4.3}$$

Equations (1.4.1) and (1.4.2) imply

$$H(t)E(-t) = 1. \tag{1.4.4}$$

1.5. Complete Symmetric Functions in Terms of Power Sums.

LEMMA. *We have*

$$H(t) = \sum_{\alpha \in \mathrm{Par}} \frac{p_\alpha t^{|\alpha|}}{z_\alpha}.$$

PROOF. We have

$$\begin{aligned}
\frac{p_\alpha t^{|\alpha|}}{z_\alpha} &= \sum_{a_i \in \mathbf{N}} \prod_{i \in \mathbf{P}} \frac{p_i^{a_i} t^{i a_i}}{i^{a_i} a_i!} \\
&= \prod_{i \in \mathbf{P}} \sum_{a_i \in \mathbf{N}} \frac{(p_i t^i / i)^{a_i}}{a_i!} \\
&= \prod_{i \in \mathbf{P}} \exp(p_i t / i) \\
&= \exp\left(\sum_{j \in \mathbf{P}} \sum_{i \in \mathbf{P}} \frac{(x_j t)^i}{i} \right) \\
&= \prod_{j \in \mathbf{P}} \exp\left(\log \frac{1}{1 - x_j t} \right) \\
&= \prod_{j \in \mathbf{P}} \frac{1}{1 - x_j t} = H(t),
\end{aligned}$$

as claimed. □

The expansion of h_n in terms of power sum symmetric functions can be obtained by comparing coefficient of t^n on both sides of the identity of the lemma:

$$h_n = \sum_{\lambda \in \mathrm{Par}_n} \frac{p_\lambda}{z_\lambda}. \tag{1.5.1}$$

1.6. Schur Functions. For each partition $\lambda = (\lambda_1, \dots, \lambda_l)$ the Schur function s_λ can be defined by the $l \times l$ Jacobi-Trudi determinants

$$s_\lambda = \det(h_{\lambda_i + j - i}) = \det(e_{\lambda'_i + j - i})$$

Here λ' denotes the partition conjugate to λ as in § 1.1, and it is understood that $h_i = 0$ and $e_i = 0$ for $i < 0$. An easy way to remember how to write down a Jacobi-Trudi determinant is to first write $h_{\lambda_1}, \dots, h_{\lambda_l}$ along the principal diagonal and then note that the subscripts increment along each row from left to right. For example,

$$s_{(2,2,1)} = \det \begin{pmatrix} h_2 & h_3 & h_4 \\ h_1 & h_2 & h_3 \\ 0 & h_0 & h_1 \end{pmatrix}.$$

The conjugate of the partition $(2, 2, 1)$ is $(3, 2)$, and the corresponding determinant of elementary symmetric functions is

$$s_{(2,2,1)} = \det \begin{pmatrix} e_3 & e_4 \\ e_1 & e_2 \end{pmatrix}.$$

The symmetric functions $\{s_\lambda \mid \lambda \in \mathrm{Par}_d\}$ also form a basis of Λ^d for every $d \in \mathbf{N}$.

1.7. The Hall Inner Product. The Schur inner product on Λ^d is the one with respect to which Schur functions form an orthogonal basis:

$$(s_\lambda, s_\mu)_n = \delta_{\lambda\mu} \text{ for all } \lambda, \mu \in \mathrm{Par}_n\,.$$

We have (see [2, §1.6])

$$(h_\lambda, m_\mu)_n = \delta_{\lambda\mu}, \tag{1.7.1}$$

$$(p_\lambda, p_\mu)_n = z_\lambda \delta_{\lambda\mu}. \tag{1.7.2}$$

1.8. Pieri Rules. The elementary and complete symmetric functions e_n and h_n are in fact special cases of Schur functions:

$$s_{(n)} = h_n \text{ and } s_{1^n} = e_n \text{ for all } n \in \mathbf{N}.$$

Pieri rules [4, Section 6] allow us to multiply an arbitrary Schur function by one of these, and express the result as a sum of Schur functions.

DEFINITION. Given partitions λ and μ, we say that $\mu - \lambda$ is a horizontal (vertical) strip of length k if $|\mu| = |\lambda| + k$, the Young diagram of μ contains the Young diagram of λ, and every column (row) of the Young diagram of μ contains at most one cell that is not in the Young diagram of λ.

LEMMA. *Suppose that $\lambda = (\lambda_1, \ldots, \lambda_l)$ and $\mu = (\mu_1, \ldots, \mu_m)$. Then $\mu - \lambda$ is horizontal strip if and only if $m - 1 \leq l \leq m$ and*

$$\mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \cdots \geq \mu_{l-1} \geq \lambda_{l-1} \geq \mu_m \geq \lambda_m, \tag{1.8.1}$$

taking $\lambda_m = 0$ if $l = m - 1$.

THEOREM (Pieri Rules). *For every $\lambda \in \mathrm{Par}$,*

$$s_\lambda h_k = \sum_\mu s_\mu, \tag{1.8.2}$$

the sum being over all partitions μ such that $\lambda - \mu$ is a horizontal strip of length k, and

$$s_\lambda e_k = \sum_\mu s_\mu, \tag{1.8.3}$$

the sum being over all partitions μ such that $\lambda - \mu$ is a vertical strip of length k.

1.9. An Inversion Formula. If $\mu - \lambda$ is a horizontal (vertical) strip we write "$\mu \in \lambda +$ h.s." or "$\lambda \in \mu -$ h.s." ("$\mu \in \lambda +$ v.s." or "$\lambda \in \mu -$ v.s.").

LEMMA. *Suppose f and g are $\mathbf{Q}$-valued functions on Par. Then*

$$g(\lambda) = \sum_{\mu \in \lambda - \text{ h.s.}} f(\mu) \tag{1.9.1}$$

if and only if

$$f(\lambda) = \sum_{\mu \in \lambda - \text{ v.s.}} (-1)^{|\lambda|-|\mu|} g(\mu). \tag{1.9.2}$$

PROOF. Define $F, G \in \Lambda[[t]]$ by

$$F(t) = \sum_{\lambda \in \mathrm{Par}} t^{|\lambda|} f(\lambda) s_\lambda, \quad G(t) = \sum_{\lambda \in \mathrm{Par}} t^{|\lambda|} g(\lambda) s_\lambda.$$

Then

$$\begin{aligned} F(t)H(t) &= \sum_{\lambda \in \mathrm{Par}} \sum_{n \in \mathbf{N}} t^{n+|\lambda|} f(\lambda) s_\lambda h_n \\ &= \sum_{\lambda \in \mathrm{Par}} \sum_{\mu \in \lambda + \text{ h.s.}} t^{|\mu|} s_\mu && \text{by (1.8.2)} \\ &= \sum_{\mu \in \mathrm{Par}} t^{|\mu|} s_\mu \sum_{\lambda \in \mu - \text{ h.s.}} f(\lambda). \end{aligned}$$

Thus (1.9.1) is equivalent to the assertion that $F(t)H(t) = G(t)$. Similarly,

$$G(t)E(-t) = \sum_{\mu \in \mathrm{Par}} t^{|\mu|} s_\mu \sum_{\lambda \in \mu - \text{ v.s.}} (-1)^{|\mu|-|\lambda|} g(\lambda).$$

Thus (1.9.2) is equivalent to the assertion that $G(t)E(-t) = F(t)$. The equivalence of $F(t)H(t) = G(t)$ and $G(t)E(-t) = F(t)$ is a consequence of (1.4.4). $\square$

1.10. The Frobenius Characteristic.

DEFINITION. Given a class function $f : S_n \to \mathbf{Q}$, its Frobenius characteristic is defined as

$$\mathrm{ch}_n f = \frac{1}{n!} \sum_{w \in S_n} f(w) p_{\alpha_w},$$

where α_w denotes the cycle type of the permutation w.

In view of § 1.2,

$$\mathrm{ch}_n f = \sum_{\alpha \vdash n} \frac{1}{z_\alpha} f(w_\alpha) p_\alpha,$$

where, for each $\alpha \in \mathrm{Par}_n$, w_α denotes a permutation with cycle type α.

The Frobenius characteristic takes the Schur inner product on class functions (§ 1.3) on S_n to the Hall inner product on Λ^n (§ 1.7):

THEOREM. *For class functions χ and η on S_n,*

$$\langle \chi, \eta \rangle_n = (\mathrm{ch}_n \chi, \mathrm{ch}_n \eta)_n.$$

PROOF. We have

$$\langle \chi, \eta \rangle_n = \sum_{\alpha \vdash n} \frac{1}{z_\alpha} \chi(w_\alpha) \eta(w_\alpha)$$

On the other hand, using the orthogonality of the powersum symmetric functions from 1.7,

$$\begin{aligned} (\mathrm{ch}_n \chi, \mathrm{ch}_n \eta)_n &= \left(\sum_{\alpha \vdash n} \frac{1}{z_\alpha} \chi(w_\alpha) p_\alpha, \sum_{\alpha \vdash n} \frac{1}{z_\alpha} \eta(w_\alpha) p_\alpha \right)_n \\ &= \sum_{\alpha \vdash n} \frac{1}{z_\alpha} \chi(w_\alpha) \eta(w_\alpha), \end{aligned}$$

thereby proving the theorem. □

1.11. The Murnaghan-Nakayama Formula. The irreducible characters of S_n are $\{\chi^\lambda \mid \lambda \in \mathrm{Par}_n\}$. They are characterized by

$$\mathrm{ch}_n \chi^\lambda = s_\lambda. \qquad (1.11.1)$$

The identity (1.11.1) is in fact equivalent to the Murnaghan-Nakayama formula for the expansion of a Schur function in the power-sum basis [5, Section 5.4]:

$$s_\lambda = \mathrm{ch}_n \chi^\lambda$$

(1.11.2) $$= \sum_{\alpha \in \mathrm{Par}_n} \chi_\lambda(w_\alpha) \frac{p_\alpha}{z_\alpha}.$$

Inverting this identity gives

(1.11.3) $$p_\alpha = \sum_{\lambda \in \mathrm{Par}_n} \chi_\lambda(w_\alpha) s_\lambda \text{ for all } \alpha \in \mathrm{Par}_n.$$

2. Character Polynomials and their Frobenius Characteristics

2.1. Character Polynomials.

Let

$$Q = \mathbf{Q}[X_1, X_2, \dots],$$

the ring of polynomials in infinitely many variables $X_1, X_2, \dots$. The ring Q has basis

$$\mathcal{B}_{\mathrm{mon}} = \{X^\alpha \mid \alpha \in \mathrm{Par}\},$$

where for each partition $\alpha = 1^{a_1} 2^{a_2} \cdots$,

$$X^\alpha = \prod_{i \in \mathbf{P}} X_i^{a_i}.$$

The variable X_i is considered to be homogeneous of degree i so that, if $\alpha \in \mathrm{Par}_n$, then the monomial X^α is homogeneous of degree n. The basis $\mathcal{B}_{\mathrm{mon}}$ will be called the *monomial basis* of Q.

For each $i \in \mathbf{P}$, $n \in \mathbf{N}$ and $w \in S_n$, let $X_i(w)$ denote the number of i-cycles in w. Thus X_i can be regarded as a class function $X_i : S_n \to \mathbf{Q}$ for every $n \in \mathbf{N}$. It follows that every $q \in Q$ can also be regarded as a class function $q : S_n \to \mathbf{Q}$ for every $n \in \mathbf{N}$.

EXAMPLE. For each $n \in \mathbf{N}$, consider the representation of S_n on $\mathbf{Q}^n$ where $w \in S_n$ acts by the corresponding permutation matrix. Then

$$X_1(w) = \mathrm{trace}(w; \mathbf{Q}^n) \text{ for every } n \in \mathbf{N}.$$

DEFINITION. Let $\{V_n\}_{n \in \mathbf{N}}$ be a family of representations with V_n a representation of S_n. We say that the family $V = \{V_n\}$ has polynomial character if there exists $q \in Q$ such that

(2.1.1) $$q(w) = \mathrm{trace}(w; V_n)$$

for every $n \in \mathbf{N}$. We say that $\{V_n\}$ has eventually polynomial character if (2.1.1) holds for sufficiently large n.

The family $\{\mathbf{Q}^n\}$ in the example has polynomial character given by the polynomial X_1.

2.2. Total Frobenius Characteristic. Given a family $f = \{f_n\}$, where $f_n : S_n \to \mathbf{Q}$ is a class function, define its *total Frobenius characteristic* to be

$$\operatorname{ch} f = \sum_{n \in \mathbf{N}} t^n \operatorname{ch}_n f_n. \tag{2.2.1}$$

Given a family $V = \{V_d\}$ of representations, where V_d is a representation of S_d with character χ_d, define its total Frobenius characteristic $\operatorname{ch} V$ to be the total Frobenius characteristic of the family $\{\chi_d\}$ of class functions.

For example if V_d is the trivial representation of S_d for each d, then $\operatorname{ch} V = H(t)$, and if V_d is the sign representation, then $\operatorname{ch} V = E(t)$.

2.3. Induction. The product structure in Λ corresponds to induction of representations of symmetric groups in the following sense:

THEOREM. *Suppose χ is a character of S_m and η a character of S_n. Then*

$$\operatorname{ch}_m \chi \operatorname{ch}_n \eta = \operatorname{ch}_{m+n} \operatorname{Ind}_{S_m \times S_n}^{S_{m+n}} (\chi \otimes \eta).$$

In particular, if $\lambda \in \operatorname{Par}_n$ then

$$\operatorname{ch}_{m+n} \operatorname{Ind}_{S_n \times S_k}^{S_{n+k}} (\chi_\lambda \otimes \chi_{(k)}) = s_\lambda h_k. \tag{2.3.1}$$

Therefore, by the Pieri rule (1.8.2),

$$\operatorname{Ind}_{S_n \times S_k}^{S_{n+k}} (\chi_\lambda \otimes \chi_{(k)}) = \bigoplus_{\mu \in \lambda + \text{ h.s.}} \chi_\mu. \tag{2.3.2}$$

2.4. The Binomial Basis. For each partition $\alpha = 1^{a_1} 2^{a_2} \cdots$ define

$$\binom{X}{\alpha} = \prod_{i \in \mathbf{P}} \binom{X_i}{a_i}.$$

Then $\binom{X}{\alpha} = X^\alpha +$ lower degree terms. It follows that

$$\mathcal{B}_{\text{bin}} = \left\{ \binom{X}{\alpha} \mid \alpha \in \operatorname{Par} \right\}$$

is a basis of Q.

2.5. Total Frobenius Characteristic of a Character Polynomial.

THEOREM. *For every* $\alpha \in \mathrm{Par}$*,*

$$\mathrm{ch}\binom{X}{\alpha} = \frac{p_\alpha t^{|\alpha|}}{z_\alpha} H(t).$$

PROOF. For partitions $\alpha = 1^{a_1}2^{a_2}\cdots$ and $\beta = 1^{b_1}2^{b_2}\cdots$, let $\binom{\beta}{\alpha} = \prod_i \binom{b_i}{a_i}$. Observe that $\binom{\beta}{\alpha} = 0$ unless $b_i \geq a_i$ for all i. We have

$$\begin{aligned}\mathrm{ch}\binom{X}{\alpha} &= \sum_{n\in\mathbf{N}}\sum_{\beta\vdash n}\binom{\beta}{\alpha}\frac{p_\beta}{z_\beta}t^n \\ &= \sum_{b_i\in\mathbf{N}}\prod_{i\in\mathbf{P}}\frac{b_i!}{a_i!(b_i-a_i)!}\frac{p_i^{b_i}}{i^{b_i}b_i!}t^{ib_i} \\ &= \prod_{i\in\mathbf{P}}\frac{p_i^{a_i}t^{ia_i}}{i^{a_i}a_i!}\sum_{b_i\geq a_i}\prod_{i\in\mathbf{P}}\frac{p_i^{b_i-a_i}t^{i(b_i-a_i)}}{i^{b_i-a_i}(b_i-a_i)!} \\ &= \frac{p_\alpha t^{|\alpha|}}{z_\alpha}H(t),\end{aligned}$$

using §1.5. □

2.6. Image of Total Frobenius Characteristic.

THEOREM. *The image of character polynomials under the total Frobenius characteristic is given by*

$$\mathrm{ch}(Q) = \Lambda[[t]]H(t).$$

PROOF. By § 2.5, $\mathrm{ch}\binom{X}{\alpha} \in \Lambda[[t]]H(t)$ for every $\alpha \in \mathrm{Par}$. Since these elements form the basis $\mathcal{B}_{\mathrm{bin}}$ of Q, it follows that $\mathrm{ch}(Q) \subset \Lambda[[t]]H(t)$. For the converse, if $\phi(t) \in \Lambda[[t]]H(t)$, by (1.4.4), $\phi(t)E(-t) \in \Lambda[[t]]$. Therefore, we may write

$$\phi(t)E(-t) = \sum_\alpha c_\alpha \frac{p_\alpha}{z_\alpha}t^{|\alpha|}$$

for some rational coefficients $\{c_\alpha\}_{\alpha\in\mathrm{Par}}$. It follows that

$$\phi = \sum_\alpha c_\alpha \frac{p_\alpha}{z_\alpha}t^{|\alpha|}H(t) = \mathrm{ch}\left(\sum_{\alpha\in\mathrm{Par}} c_\alpha \binom{X}{\alpha}\right),$$

whence $\phi \in \mathrm{ch}(Q)$. □

2.7. Stabilized Schur Functions. The results of § 2.3 and 2.6 suggest the following construction for representations with polynomial character: For every $\lambda \in \mathrm{Par}_n$ and every $m \in \mathbf{N}$, let

$$\rho_\lambda^m = \sum_{\mu \vdash m,\, \mu \in \lambda + \text{h.s.}} \chi_\mu. \tag{2.7.1}$$

In particular, if $\rho_\lambda^m = \chi_\lambda$ if $m = |\lambda|$, and $\rho_\lambda^m = 0$ if $m < |\lambda|$. Then the total Frobenius characteristic of this family of characters is

$$\mathrm{ch}\{\rho_\lambda^m\} = t^{|\lambda|} s_\lambda H(t).$$

We call $t^{|\lambda|} s_\lambda H(t)$ the *stabilized Schur function*. Since the Schur functions gives rise to a basis

$$\{t^{|\lambda|} s_\lambda \mid \lambda \in \mathrm{Par}\}$$

of $\Lambda[[t]]$, the stabilized Schur functions

$$\{t^{|\lambda|} s_\lambda H(t) \mid \lambda \in \mathrm{Par}\}$$

form a basis of $\Lambda[[t]]H(t)$. However, the homogeneous components of $t^{|\lambda|} s_\lambda H(t)$ are not, in general, Frobenius characteristics of irreducible representations.

EXAMPLE. Taking $\lambda = (1)$, $\rho_{(1)}^m = \mathrm{Ind}_{S_1 \times S_{m-1}}^{S_m} \chi_{(1)} \otimes \chi_{(m-1)} = \chi_{(m)} + \chi_{(m-1,1)}$. Thus $\rho_{(1)}^m$ is the representation of S_m on $\mathbf{Q}^m$ by permutation matrices.

Using the expansion of s_λ in terms of power sum symmetric functions from § 1.11, we have

$$t^{|\lambda|} s_\lambda H(t) = \sum_{\alpha \vdash |\lambda|} \chi_\lambda(w_\alpha) \frac{t^{|\alpha|} p_\alpha}{z_\alpha} H(t) = \sum_{\alpha \vdash |\lambda|} \chi_\lambda(w_\alpha)\, \mathrm{ch} \binom{X}{\alpha}.$$

THEOREM. *For every $\lambda \in \mathrm{Par}$ and $m \geq |\lambda|$, let ρ_λ^m be as in (2.7.1). The family $\rho_\lambda = \{\rho_\lambda^m\}$ has polynomial character given by*

$$u_\lambda = \sum_{\alpha \vdash |\lambda|} \chi_\lambda(w_\alpha) \binom{X}{\alpha}. \tag{2.7.2}$$

COROLLARY. *The set*

$$\mathcal{B}_{\text{stab}} = \{u_\lambda \mid \lambda \in \mathrm{Par}\}$$

is a basis of Q.

2.8. Padded Partitions. Given $\lambda = (\lambda_1, \dots, \lambda_l) \in \mathrm{Par}$ and $n \in \mathbf{N}$ such that $n \geq |\lambda| + \lambda_1$, define $\lambda[n] \in \mathrm{Par}_n$ by

$$\lambda[n] = (n - |\lambda|, \lambda_1, \dots, \lambda_l). \tag{2.8.1}$$

For example, if $\lambda = \emptyset$, the empty partition, then $\lambda[n] = (n)$ is defined for all $n \geq 0$.

For a partition $\mu = (\mu_1, \dots, \mu_m)$, let μ^+ denote the partition $(\mu_2, \dots, \mu_m)$.

LEMMA. *For $\lambda = (\lambda_1, \dots, \lambda_l)$ and $\mu = (\mu_1, \dots, \mu_m) \in \mathrm{Par}$, $\mu - \lambda$ is a horizontal strip if and only if $\mu_1 \geq \lambda_1$ and $\lambda - \mu^+$ is a horizontal strip.*

PROOF. This is an easy consequence of Lemma 1.8. □

COROLLARY. *Given partitions λ and μ and an integer n, $\mu[n] - \lambda$ is a horizontal strip if and only if $n \geq \lambda_1 + |\mu|$ and $\lambda - \mu$ is a horizontal strip.*

We have

$$\begin{aligned} t^{|\lambda|} s_\lambda H(t) &= \sum_{n \geq |\lambda|} t^n \sum_{\mu \vdash n,\ \mu \in \lambda + \text{ h.s.}} s_\mu \\ &= \sum_{n \geq |\lambda|} t^n \sum_{\lambda \in \mu^+ + \text{ h.s.}} s_{\mu^+[n]} \\ &= \sum_{\mu \in \lambda - \text{ h.s.}} \sum_{n \geq |\lambda|} t^n s_{\mu[n]}. \end{aligned} \tag{2.8.2}$$

Define

$$\tilde{s}_\lambda = \sum_{n \geq \lambda_1 + |\lambda|} t^n s_{\lambda[n]}.$$

Then (2.8.2) can be interpreted as

$$t^{|\lambda|} s_\lambda H(t) \equiv \sum_{\mu \in \lambda - \text{ h.s.}} \tilde{s}_\mu. \tag{2.8.3}$$

By Lemma 1.9, we get

$$\tilde{s}_\lambda = \sum_{\mu \in \lambda - \text{ v.s.}} (-1)^{|\lambda| - |\mu|} t^{|\mu|} s_\mu H(t). \tag{2.8.4}$$

2.9. Specht Character Polynomials. Using (2.7.2) and (2.8.4), we have:

THEOREM. *For every partition λ, the family $\{s_{\lambda[n]}\}$ has eventually polynomial character given by*

$$q_\lambda = \sum_{\mu \in \lambda - \text{ v.s.}} (-1)^{|\lambda| - |\mu|} \sum_{\alpha \vdash |\mu|} \chi_\mu(w_\alpha) \binom{X}{\alpha}. \tag{2.9.1}$$

The polynomial q_λ is called the Specht character polynomial.

COROLLARY. *The set*

$$\mathcal{B}_{\text{Specht}} = \{q_\lambda \mid \lambda \in \text{Par}\}$$

is a basis of Q.

2.10. Multiplicity of a Specht module.

THEOREM. *For $\lambda \in \text{Par}_n$, let χ_λ denote the irreducible character of S_n corresponding to λ. Then, for all $\beta \in \text{Par}$,*

$$\left\langle \binom{X}{\beta}, \chi_\lambda \right\rangle_n = \sum_{\alpha \vdash |\beta|,\ \alpha \in \lambda - \textit{h.s.}} \frac{\chi_\alpha(w_\beta)}{z_\beta}.$$

PROOF. Since the Frobenius characteristic maps the Schur inner product to the Hall inner product (§ 1.10), by Theorem 2.5 and Pieri's rule (1.8.2),

$$\left\langle \binom{X}{\beta}, \chi_\lambda \right\rangle_n = \left(\frac{p_\beta}{z_\beta} h_{|\lambda|-|\beta|}, s_\lambda \right)_n.$$

Applying the expansion (1.11.3) of p_β in terms of Schur functions, we get

$$\left\langle \binom{X}{\beta}, \chi_\lambda \right\rangle_n = \frac{1}{z_\beta} \sum_{|\alpha| \vdash |\beta|} \chi_\alpha(w_\beta)(s_\alpha h_{|\lambda|-|\beta|}, s_\lambda)_n.$$

The Schur inner product in the above expression can be evaluated using Pieri's rule (1.8.2) to give the desired identity. □

2.11. Moments and Signed Moments. Given $q \in Q$ and $n \in \mathbf{N}$, define the nth moment $\langle q \rangle_n$ and nth signed moment $\{q\}_n$ of q by

$$\langle q \rangle_n = \frac{1}{n!} \sum_{w \in S_n} q(w)$$

$$\{q\}_n = \frac{1}{n!} \sum_{w \in S_n} \epsilon(w) q(w).$$

THEOREM. *For every partition α,*

$$\left\langle \binom{X}{\alpha} \right\rangle_n = \begin{cases} \frac{1}{z_\alpha} & \textit{if } n \geq |\alpha|, \\ 0 & \textit{otherwise.} \end{cases}$$

$$\left\{ \binom{X}{\alpha} \right\}_n = \begin{cases} \frac{\text{sgn}(w_\alpha)}{z_\alpha} & \textit{if } n \in \{|\alpha|-1, |\alpha|\}, \\ 0 & \textit{otherwise.} \end{cases}$$

PROOF. Take $\lambda = (n)$ and $\lambda = (1^n)$, respectively in Theorem 2.10. □

3. Representations of GL_n

3.1. Polynomial Representations. For a $\mathbf{Q}$-vector space V, let V^* denote its linear dual.

DEFINITION. A homogeneous polynomial representation of $\mathrm{GL}_m(\mathbf{Q})$ of degree n consists of a $\mathbf{Q}$-vector space V, together with a group homomorphism $\rho : \mathrm{GL}_m(\mathbf{Q}) \to \mathrm{GL}_{\mathbf{Q}}(V)$ such that, for every $v \in V$ and $\xi \in V^*$, $g \mapsto \xi(\rho(g)v$ is homogeneous polynomial function of degree n of the entries of the matrix $g \in M_m(\mathbf{Q})$.

Equivalently, the matrix of $\rho(g)$ with respect to any basis of V has entries that are polynomial functions of the entries of g.

EXAMPLE. Let $\{\mathbf{e}_1, \dots, \mathbf{e}_m\}$ denote the basis of coordinate vectors of $\mathbf{Q}^m$. The space $V = \mathrm{Sym}^2 \mathbf{Q}^m$ of symmetric tensors of degree two has a basis $\{\mathbf{e}_i\mathbf{e}_j \mid 1 \leq i \leq j \leq n\}$. If $g \in \mathrm{GL}_m(\mathbf{Q})$ has matrix (g_{ij}), then

$$\begin{aligned} g \cdot (\mathbf{e}_i\mathbf{e}_j) &= \left(\sum_{k=1}^n g_{ki}\mathbf{e}_k\right)\left(\sum_{l=1}^n g_{lj}\mathbf{e}_l\right) \\ &= \sum_{k=1}^n g_{ki}g_{kj}\mathbf{e}_k^2 + \sum_{1\leq k<l\leq n} 2g_{ki}g_{lj}\mathbf{e}_k\mathbf{e}_l \end{aligned}$$

Therefore $\mathrm{Sym}^2 \mathbf{Q}^m$ is a homogeneous polynomial representation of $\mathrm{GL}_m(\mathbf{Q})$ degree 2.

EXAMPLE. The exterior square $V = \wedge^2\mathbf{Q}^m$ has a basis $\{\mathbf{e}_i \wedge \mathbf{e}_j \mid 1 \leq i < j \leq n\}$. If $g \in \mathrm{GL}_m(\mathbf{Q})$ has matrix (g_{ij}), then

$$\begin{aligned} g \cdot (\mathbf{e}_i \wedge \mathbf{e}_j) &= \left(\sum_{k=1}^n g_{ki}\mathbf{e}_k\right) \wedge \left(\sum_{l=1}^n g_{lj}\mathbf{e}_l\right) \\ &= \sum_{1\leq k<l\leq n} (g_{ki}g_{lj} - g_{lj}g_{ki})\mathbf{e}_k\mathbf{e}_l \end{aligned}$$

Therefore $\wedge^2\mathbf{Q}^m$ is a homogeneous polynomial representation of $\mathrm{GL}_m(\mathbf{Q})$ degree 2.

In general $\mathrm{Sym}^n \mathbf{Q}^m$ and $\wedge^n\mathbf{Q}^m$ are homogeneous polynomial representations of $\mathrm{GL}_m(\mathbf{Q})$ degree n.

3.2. The Character of a Polynomial Representation.

DEFINITION. If (ρ, V) is a homogeneous polynomial representation of $GL_m(\mathbf{Q})$ of degree n, its character is defined as

$$\mathrm{char}\, \rho(x_1, \dots, x_m) = \mathrm{trace}(\rho(\mathrm{diag}(x_1, \dots, x_m)); V),$$

where $\operatorname{diag}(x_1, \dots, x_m)$ denotes the diagonal matrix in $\mathrm{GL}_m(\mathbf{Q})$ with diagonal entries $x_1, \dots, x_m$.

For any $w \in S_m$, $\operatorname{diag}(x_1, \dots, x_m)$ is conjugate to $\operatorname{diag}(x_{w(1)}, \dots, x_{w(m)})$ in $GL_m(\mathbf{Q})$. It follows that the polynomial $\operatorname{char} \rho(x_1, \dots, x_m)$ is symmetric in the variables $x_1, \dots, x_m$.

3.3. Character of symmetric and exterior tensors. Given a symmetric function $f(x_1, x_2, \dots)$ as in § 1.4, its specialization of m variables is the symmetric polynomial in m variables obtained by setting $x_{m+1} = x_{m+2} = \dots = 0$. We will denote this specialization by $f(x_1, \dots, x_m)$.

THEOREM. *For every non-negative integer n, it is easy to see that*

$$\operatorname{char} \operatorname{Sym}^n \mathbf{Q}^m = h_n(x_1, \dots, x_m),$$
$$\operatorname{char} \wedge^n \mathbf{Q}^n = e_n(x_1, \dots, x_m).$$

PROOF. $Sym^n \mathbf{Q}^m$ has basis

$$\{e_{i_1} \cdots e_{i_n} \mid 1 \leq i_1 \leq \cdots \leq i_n \leq m\}.$$

The diagonal matrix $\operatorname{diag}(x_1, \dots, x_m)$ scales basis vector $e_{i_1} \cdots e_{i_n}$ by a factor of $x_{i_1} \cdots x_{i_m}$. Thus the trace of $\operatorname{diag}(x_1, \dots, x_m)$ on $\operatorname{Sym}^n \mathbf{Q}^m$ is

$$h_n(x_1, \dots, x_m) = \sum_{1 \leq i_1 \leq \cdots \leq i_n \leq m} x_{i_1} \cdots x_{i_n}.$$

The proof for $\wedge^n \mathbf{Q}^m$ is similar. $\wedge^n \mathbf{Q}^m$ has basis

$$\{e_{i_1} \cdots e_{i_n} \mid 1 \leq i_1 < \cdots < i_n \leq m\},$$

whence the trace of $\operatorname{diag}(x_1, \dots, x_m)$ on $\operatorname{Sym}^n \mathbf{Q}^m$ is

$$e_n(x_1, \dots, x_m) = \sum_{1 \leq i_1 M \cdots < i_n \leq m} x_{i_1} \cdots x_{i_n}.$$

□

3.4. Irreducible Polynomial Representations of $\mathrm{GL}_m(\mathbf{Q})$. A virtual polynomial representation of $\mathrm{GL}_m(\mathbf{Q})$ is a formal difference $U - V$ of polynomial representations. In the spirit of the construction of integers from natural numbers, formal differences $U - V$ and $U' - V'$ are considered to be the same if $U \oplus V'$ is isomorphic to $U' \oplus V$.

A polynomial representation (ρ, V) of $\mathrm{GL}_m(\mathbf{Q})$ is said to be irreducible if it does not admit any proper non-trivial invariant subspace. It is a famous result of Schur that every polynomial representation of $GL_m(\mathbf{Q})$ can be written uniquely as a direct sum of irreducible polynomial representations, and that every irreducible polynomial representation is homogeneous.

Thus the set of virtual representations of $\mathrm{GL}_m(\mathbf{Q})$ is the free Abelian group generated by irreducible homogeneous polynomial representations.

This Abelian group can be regarded as a ring, with product structure coming from the tensor product of representations.

It follows from the definition of char (§ 3.2) that, for polynomial representations V and W of $\mathrm{GL}_m(\mathbf{Q})$,

$$\mathrm{char}\, V \otimes W = \mathrm{char}\, V\, \mathrm{char}\, W.$$

Note that the tensor product of homogeneous polynomial representations of degree n_1 and n_2 is $n_1 + n_2$.

EXAMPLE. Consider the defining representation of $\mathrm{GL}_m(\mathbf{Q})$ on $\mathbf{Q}^m$. This is an irreducible homogeneous polynomial of degree 1, and

$$\mathrm{char}\, \mathbf{Q}^n = x_1 + \cdots + x_m = h_1(x_1, \ldots, x_m).$$

Furthermore, the character of its d-fold tensor power is given by

$$\mathrm{char}(\mathbf{Q}^n)^{\otimes d} = (x_1 + \cdots + x_m)^d.$$

3.5. Constructing Irreducible Representations of $\mathrm{GL}_m(\mathbf{Q})$. Let $\lambda = (\lambda_1, \cdots, \lambda_m)$ be a partition of n with at most m positive parts. Using the operations of addition and multiplication on the ring of virtual polynomial representations of $\mathrm{GL}_m(\mathbf{Q})$, define the *Weyl module* associated to λ by a Jacobi-Trudi type (see § 1.6) determinants

$$W_\lambda(\mathbf{Q}^m) = \det(\mathrm{Sym}^{\lambda_i + j - i}\, \mathbf{Q}^m) = \det(\wedge^{\lambda'_i + j - i}).$$

Since determinants involve positive and negative terms, the above expressions are potentially virtual representations that turn out to be irreducible representations after cancellation.

THEOREM. *The representations $W_\lambda(\mathbf{Q}^m)$, as λ runs over the set of integer partitions of n with at most m parts, form a complete set of representatives of the homogeneous irreducible polynomial representations of $\mathrm{GL}_m(\mathbf{Q})$ of degree n. Moreover,*

$$\mathrm{char}\, W_\lambda(\mathbf{Q}^m) = s_\lambda(x_1, \ldots, x_m).$$

3.6. Counting Subsets and Multisets. A weighted set is a set A, together with a *weight function* $v : A \to \mathbf{P}$. Given a subset $B \subset A$, define the weight of B as

$$v(B) = \sum_{b \in B} v(b).$$

A multiset B with elements drawn from A is a function $f_B : A \to \mathbf{N}$, where, for each $a \in A$, $f_B(a)$ can be thought of as the multiplicity of a in B. We write $B \sqsubset A$. Given $B \sqsubset A$, the weight of B is defined as

$$v(B) = \sum_{a \in A} f_B(a) v(a).$$

In a set with no specified weight function, it is customary to assume $v(a) = 1$ for every $a \in A$. In that case, the weight of a subset is its cardinality.

THEOREM 3.1. *Let (A, v) be a weighted set. Let $\binom{A}{n}$ denote the number of subsets of A with weight n, and let $\left[{A \atop n}\right]$ denote the number of multisets with elements drawn from A with weight n. Then*

$$\sum \binom{A}{n} t^n = \prod_{a\in A}(1 + t^{v(a)})$$

$$\sum \begin{bmatrix} A \\ n \end{bmatrix} t^n = \prod_{a\in A}(1 - t^{v(a)})^{-1}.$$

In particular, if, for each $i \in \mathbf{P}$, a_i is the number of elements of A with weight i, then

$$\sum \binom{A}{n} t^n = \prod_{i\in \mathbf{P}}(1 + t^{i})^{a_i}$$

$$\sum \begin{bmatrix} A \\ n \end{bmatrix} t^n = \prod_{i\in \mathbf{P}}(1 - t^{i})^{-a_i}.$$

We leave the proof as an exercise to the reader. For a nice way to think about such generating functions, see [1].

3.7. Character Polynomials of Symmetric Powers. For all $m, n \in \mathbf{N}$, $V_n = \mathrm{Sym}^m(\mathbf{Q}^n)$ is a representation of S_n.

THEOREM. *Let $\{H_m\}$ be the sequence in Q determined by the identity in $Q[[t]]$:*

$$\sum_{m\geq 0} H_m t^m = \prod_{i\geq 1}(1 - t^{i})^{-X_i}.$$

The family of representations $V_n = \mathrm{Sym}^m \mathbf{Q}^n$ (V_n is a representation of S_n) has polynomial character given by H_m for each $m \in \mathbf{N}$.

PROOF. $Sym^m(\mathbf{Q}^n)$ has a basis indexed by multisets of weight m with elements drawn from n. The trace of $w \in S_n$ on $\mathrm{Sym}^m(\mathbf{Q}^n)$ is the number of such multisets that are fixed by the action of w. These are precisely the multisets for which all the elements in each cycle of w occur with the same multiplicity. Each such multiset can therefore be regarded as a *multiset of cycles of* w, where an i-cycle has weight i. Since the number of i-cycles in w is $X_i(w)$, the result follows. □

3.8. Moment Generating Functions for Symmetric Powers.

THEOREM. *We have*

$$\sum_{m\in\mathbf{N}}\sum_{n\in\mathbf{N}}\langle H_m\rangle_n t^m v^n = \prod_{i\in\mathbf{N}}(1-vt^i)^{-1}$$
$$\sum_{m\in\mathbf{N}}\sum_{n\in\mathbf{N}}\{H_m\}_n t^m v^n = \prod_{i\in\mathbf{N}}(1+vt^i).$$

In other words, the multiplicity of the trivial representation of S_n in $\mathrm{Sym}^m\,\mathbf{Q}^n$ is the number of integer partitions of m with at most n non-zero parts, while the multiplicity of the sign representation of S_n in $\mathrm{Sym}^m\,\mathbf{Q}^n$ is the number of integer partitions of m with either n or $n-1$ distinct non-zero parts.

PROOF. The sums on the left hand sides of the formulae are over all permutations in all S_n, a partition in S_n getting weight $\frac{1}{n!}$. Such a sum can be rewritten as a sum over all integer partitions, the partition $\alpha = 1^{a_1}2^{a_2}\cdots$ getting weight $\frac{1}{z_\alpha}$. Thus we have

$$\begin{aligned}
\sum_{m\in\mathbf{N}}\sum_{n\in\mathbf{N}}\langle H_m\rangle_n t^m v^n &= \sum_{m\in\mathbf{N}}\sum_{a_i\in\mathbf{N}} H_m(a_1,a_2,\dots)t^m\prod_{i\in\mathbf{P}}\frac{v^{ia_i}}{i^{a_i}a_i!}\\
&= \sum_{a_i\in\mathbf{N}}\prod_{i\in\mathbf{P}}(1-t^i)^{-a_i}\frac{v^{ia_i}}{i^{a_i}a_i!}\\
&= \prod_{i\in\mathbf{P}}\sum_{a_i\in\mathbf{N}}\frac{1}{a_i!}\left(\frac{v^i}{i(1-t^i)}\right)^{a_i}\\
&= \prod_{i\in\mathbf{P}}\exp\left(\sum_{j\geq 0}\frac{(vt^j)^i}{i}\right)\\
&= \prod_{j\in\mathbf{N}}\exp\log\frac{1}{1-vt^j}\\
&= \prod_{j\in\mathbf{N}}\frac{1}{1-vt^j},
\end{aligned}$$

proving the first identity. For the second identity, we proceed similarly, but with a sign thrown in:

$$\begin{aligned}\sum_{m\in\mathbf{N}}\sum_{n\in\mathbf{N}}\{H_m\}_n t^m v^n &= \sum_{m\in\mathbf{N}}\sum_{a_i\in\mathbf{N}} H_m(a_1,a_2,\dots)t^m\prod_{i\in\mathbf{P}}\frac{((-1)^{i+1}v)^{ia_i}}{i^{a_i}a_i!}\\ &= \prod_{i\in\mathbf{P}}\sum_{a_i\in\mathbf{N}}\frac{1}{a_i!}\left(\frac{-(-v)^i}{i(1-t^i)}\right)^{a_i}\\ &= \prod_{i\in\mathbf{P}}\exp\left(-\sum_{j\geq 0}\frac{(-vt^j)^i}{i}\right)\\ &= \prod_{j\in\mathbf{N}}(1+vt^j),\end{aligned}$$

completing the proof. ☐

3.9. Character Polynomials of Exterior Powers. For all $m, n \in NN$, $V_n = \wedge^m(\mathbf{Q}^n)$ is a representation of S_n.

THEOREM. *Let $\{E_m\}$ be the sequence in Q determined by the identity in $Q[[t]]$:*

$$\sum_{m\geq 0} E_m t^m = \prod_{i\geq 1}(1-(-t)^i)^{X_i}.$$

The family of representations $V_n = \wedge^m\mathbf{Q}^n$ (V_n is a representation of S_n) has polynomial character given by E_m for each $m \in \mathbf{N}$.

PROOF. $\wedge^m(\mathbf{Q}^n)$ has a basis indexed by subsets of $[n]$ of cardinality m. The trace of $w \in S_n$ on $\wedge^m(\mathbf{Q}^n)$ is the signed sum of such subsets that are fixed by the action of w. These are precisely the unions of cycles of w. Each such subset can therefore be regarded as a *set of cycles of* w, where an i-cycle has weight i. The sign of an i-cycle is $-(-1)^i$. Since the number of i-cycles in w is $X_i(w)$, the result follows. ☐

3.10. Moment Generating Function for Exterior Powers.

THEOREM. *We have*

$$\sum_{m\in\mathbf{N}}\sum_{n\in\mathbf{N}}\langle E_m\rangle_n t^m v^n = \frac{1+tv}{1-v},$$

$$\sum_{m\in\mathbf{N}}\sum_{n\in\mathbf{N}}\{E_m\}_n t^m v^n = \frac{1+v}{1-tv}.$$

In other words, the trivial representation of S_n occurs with multiplicity one in $\wedge^m\mathbf{Q}^n$ if $m = 0$ for all $n \in \mathbf{N}$, or if $m = 1$ and $n \in \mathbf{P}$, and does not occur otherwise. The sign representation of S_n occurs with multiplicity one *in* $\wedge^m \mathbf{Q}^n$ *if and only if* $n = m \geq 0$ *or* $n = m+1 \geq 0$*, and does not occur otherwise.*

PROOF. Using § 3.10 and proceeding as in § 3.8, we get

$$\begin{aligned}\sum_{m\in\mathbf{N}}\sum_{n\in\mathbf{N}} \langle E_m\rangle_n\, t^m v^n &= \prod_{i\in\mathbf{P}}\sum_{a_i\in\mathbf{N}} \frac{1}{a_i!}\left(\frac{v^i(1-(-t)^i)}{i}\right)^{a_i}\\ &= \prod_{i\in\mathbf{P}} \exp\left(\frac{v^i}{i} - \frac{(-vt)^i}{i}\right)\\ &= \frac{1+vt}{1-v},\end{aligned}$$

as claimed. The proof of the second identity is similar. □

3.11. The General Moment Generating Function. For each $\mathbf{x} = (x_1, \ldots, x_l) \in \mathbf{N}^l$, let

$$\begin{aligned}H_{\mathbf{x}} &= H_{x_1}\cdots H_{x_l},\\ E_{\mathbf{x}} &= E_{x_1}\cdots E_{x_l}.\end{aligned}$$

THEOREM 3.2. *For all* $l, m \in \mathbf{N}$*, we have*

$$\sum_{\mathbf{x}\in\mathbf{N}^l,\,\mathbf{y}\in\mathbf{N}^m,\,n\in\mathbf{N}} \langle H_{\mathbf{x}}E_{\mathbf{y}}\rangle\, t^{\mathbf{x}}u^{\mathbf{y}}v^n = \prod_{R\subset[l]}\prod_{S\subset[m]} (1-(-1)^{|S|}t^R u^S v)^{(-1)^{|S|+1}}.$$

$$\sum_{\mathbf{x}\in\mathbf{N}^l,\,\mathbf{y}\in\mathbf{N}^m,\,n\in\mathbf{N}} \{H_{\mathbf{x}}E_{\mathbf{y}}\}\, t^{\mathbf{x}}u^{\mathbf{y}}v^n = \prod_{R\subset[l]}\prod_{S\subset[m]} (1-(-1)^{|S|+1}t^R u^S v)^{(-1)^{|S|}}$$

PROOF. From § 3.7 and § 3.9 we have

$$\sum_{\mathbf{x}\in\mathbf{N}^l,\,\mathbf{y}\in\mathbf{N}^m} H_{\mathbf{x}}E_{\mathbf{y}}t^{\mathbf{x}}u^{\mathbf{y}} = \prod_{i\in\mathbf{P}}\Big(\prod_{r\in[l]}(1-t_r^i)^{-X_i}\prod_{s\in[m]}(1-(-u)_s^i)^{X_i}\Big).$$

Using the generating function and proceeding as in § 3.8 and § 3.10 we get

$$\begin{aligned}\sum_{\mathbf{x}\in\mathbf{N}^l,\,\mathbf{y}\in\mathbf{N}^m,\,n\in\mathbf{N}} \langle H_{\mathbf{x}}E_{\mathbf{y}}\rangle\, t^{\mathbf{x}}u^{\mathbf{y}}v^n &= \prod_{i\in\mathbf{P}}\sum_{a_i\in\mathbf{N}} \frac{v^{ia_i}}{i^{a_i}a_i!}\frac{\prod_{s\in[m]}(1-(-u_s)^i)^{a_i}}{\prod_{r\in[l]}(1-t_r^i)^{a_i}}\\ &= \exp\left(\sum_{i\in\mathbf{P}}\frac{v^i\prod_{s\in[m]}(1-(-u_s)^i)}{i\prod_{r\in[l]}(1-t_r^i)}\right) \qquad (3.11.1)\end{aligned}$$

Now

$$\sum_{i\in\mathbf{P}}\frac{v^i\prod_{s\in[m]}(1-(-u_s)^i)}{i\prod_{r\in[l]}(1-t_r^i)} = \sum_{R\subset[l]}\sum_{S\subset[m]}(-1)^{|S|}\sum_{i\in\mathbf{P}}\frac{((-1)^{|S|}t^R u^S v)^i}{i}$$

The formulae

$$\log\frac{1}{1-x} = \sum \frac{x^i}{i} \text{ and } \log(1+x) = -\sum \frac{(-x)^i}{i}$$

can be combined into

$$\log(1-\epsilon x)^{-\epsilon} = \epsilon \sum \frac{(\epsilon x)^i}{i} \text{ for } \epsilon = \pm 1.$$

Therefore,

$$(-1)^{|S|} \sum_{i\in\mathbf{P}} \frac{((-1)^{|S|} t^R u^S v)^i}{i} = \log(1-(-1)^{|S|} t^R u^S v)^{(-1)^{|S|+1}}$$

Using this to evaluate (3.11.1) yields the first formula in the theorem.

Similarly,

$$\sum_{\mathbf{x}\in\mathbf{N}^l,\, \mathbf{y}\in\mathbf{N}^m,\, n\in\mathbf{N}} \{H_\mathbf{x} E_\mathbf{y}\}\, t^\mathbf{x} u^\mathbf{y} v^n = \prod_{i\in\mathbf{P}} \sum_{a_i\in\mathbf{N}} \frac{-(-v)^{ia_i}}{i^{a_i} a_i!} \frac{\prod_{s\in[m]}(1-(-u_s)^i)^{a_i}}{\prod_{r\in[l]}(1-t_r^i)^{a_i}}$$

$$= \exp\left(\sum_{i\in\mathbf{P}} \frac{-(-v)^i \prod_{s\in[m]}(1-(-u_s)^i)}{i\prod_{r\in[l]}(1-t_r^i)}\right) \quad (3.11.2)$$

But now

$$\sum_{i\in\mathbf{P}} \frac{-(-v)^i \prod_{s\in[m]}(1-(-u_s)^i)}{i\prod_{r\in[l]}(1-t_r^i)} = \sum_{R\sqsubset[l]} \sum_{S\subset[m]} \sum_{i\in\mathbf{P}} (-1)^{|S|+1} \frac{((-1)^{|S|+1} t^R u^S v)^i}{i}$$

$$= \sum_{R\sqsubset[l]} \sum_{S\subset[m]} \log(1-(-1)^{|S|+1} t^R u^S)^{|S|},$$

from which the second formula follows. □

3.12. Vector Partitions and their Generating Functions. Given $\mathbf{x} \in \mathbf{N}^l$, a vector partition of $\mathbf{x}$ with n parts is a decomposition

$$\mathbf{x} = \mathbf{x}^1 + \cdots + \mathbf{x}^n, \quad (3.12.1)$$

where each $\mathbf{x}^i \in \mathbf{N}^l$, and the order in which the summands are written does not matter. Note that some of summands $\mathbf{x}^i$ are permitted to be the zero vector. We denote the number of vector partitions of $\mathbf{x}$ with n parts by $p_n(\mathbf{x})$.

THEOREM. *We have*

$$\sum_{\mathbf{x}\in\mathbf{N}^l} \sum_{n\in\mathbf{N}} p_n(\mathbf{x}) t^\mathbf{x} v^n = \prod_{R\sqsubset[l]} \frac{1}{1-t^R v}.$$

The proof is quite similar to the product decomposition of the generating function for integer partitions. A distinct part vector partition with n parts is a vector partition of the form (3.12.1) where the vectors $\mathbf{x}^1, \ldots, \mathbf{x}^n$ are all distinct integer vectors. Note that the zero vector is permitted as one of the summands in (3.12.1), but in a distinct part vector partitions, it is allowed to occur at most once. Let $q_n(\mathbf{x})$ denote the number of distinct part vector partitions of $\mathbf{x}$ with n parts.

THEOREM. *We have*

$$\sum_{\mathbf{x}\in\mathbf{N}^l} q_n(\mathbf{x}) t^{\mathbf{x}} v^n = \prod_{R\sqsubset[l]} (1 + t^R v).$$

EXAMPLE. The partitions of $(2,1)$ with two parts are given by:

$$(0,0) + (2,1),\ (1,0) + (1,1),\ (2,0) + (0,1),$$

so that $p_2(2,1) = 3$. Since all the summands in all these vector partitions are distinct, $q_2(2,1) = 3$ as well.

3.13. Vector Partitions and Symmetric Tensors.

THEOREM. *For every $l \in \mathbf{N}$,*

$$\sum_{\mathbf{x}\in\mathbf{N}^l}\sum_{n\in\mathbf{N}} \langle H_{\mathbf{x}}\rangle_n t^{\mathbf{x}} v^n = \prod_{R\sqsubset[l]} \frac{1}{1 - t^R v},$$

$$\sum_{\mathbf{x}\in\mathbf{N}^l}\sum_{n\in\mathbf{N}} \{H_{\mathbf{x}}\}_n t^{\mathbf{x}} v^n = \prod_{R\sqsubset[l]} (1 + t^R v).$$

In other words, for every $\mathbf{x} \in \mathbf{N}^l$, the multiplicity of the trivial representation of S_n in

$$\mathrm{Sym}^{\mathbf{x}}\, \mathbf{Q}^n := \mathrm{Sym}^{x_1}\, \mathbf{Q}^n \otimes \cdots \otimes \mathrm{Sym}^{x_l}\, \mathbf{Q}^n$$

is $p_n(\mathbf{x})$, and the multiplicity of the sign representation of the S_n in $\mathrm{Sym}^{\mathbf{x}}\, \mathbf{Q}^n$ is $q_n(\mathbf{x})$.

3.14. Vector Partitions with Zero-One Vectors. Given $\mathbf{y} \in \mathbf{N}^m$, let $|\mathbf{y}|$ denote the sum of the entries of $\mathbf{y}$. A vector $\mathbf{y} = (y_1, \ldots, y_l) \in \mathbf{N}^m$ is said to be a zero-one vector if $y_i \in \{0,1\}$ for $1 \leq i \leq l$. Define integer-valued functions $p_n^* : \mathbf{N}^m \to \mathbf{N}$ and $q_n^* : \mathbf{N}^m \to \mathbf{N}$ by

$$\sum_{\mathbf{y}\in\mathbf{N}^m} p_n^*(\mathbf{y}) u^{\mathbf{y}} v^n = \prod_{S\subset[m]} (1 - (-1)^{|S|} t^S v)^{(-1)^{|S|+1}}$$

$$\sum_{\mathbf{y}\in\mathbf{N}^m} q_n^*(\mathbf{y}) u^{\mathbf{y}} v^n = \prod_{S\subset[m]} (1 - (-1)^{|S|+1} t^S v)^{(-1)^{|S|}}.$$

The combinatorial interpretation of $p_n^*(\mathbf{y})$ and $q_n^*(\mathbf{y})$ are as follows:

THEOREM. *For each* $\mathbf{y} \in \mathbf{N}^m$ *and each* $n \in \mathbf{N}$, $p_n^*(\mathbf{y})$ *(resp.,* $q_n^*(\mathbf{y})$*) is the number of decompositions*

$$\mathbf{y} = \mathbf{y}^1 + \cdots + \mathbf{y}^n, \text{ where } \mathbf{y}^i \in \{0,1\}^m,$$

and the summands $\mathbf{y}^i$ *for which* $|\mathbf{y}^i|$ *is odd (resp., even) occur at most once.*

3.15. Vector Partitions and Exterior Powers.

THEOREM. *For every* $m \in \mathbf{N}$,

$$\sum_{\mathbf{y}\in\mathbf{N}^m}\sum_{n\in\mathbf{N}} \langle E_{\mathbf{y}}\rangle_n t^{\mathbf{y}} v^n = \prod_{S\subset[m]} (1-(-1)^{|S|}t^S v)^{(-1)^{|S|+1}}$$

$$\sum_{\mathbf{y}\in\mathbf{N}^m}\sum_{n\in\mathbf{N}} \{E_{\mathbf{y}}\}_n t^{\mathbf{y}} v^n = \prod_{S\subset[m]} (1-(-1)^{|S|+1}t^S v)^{(-1)^{|S|}}.$$

In other words, the multiplicity of the trivial representation of S_n *in*

$$\wedge^{\mathbf{y}}\mathbf{Q}^n := \wedge^{y_1}\mathbf{Q}^n \otimes \cdots \otimes \wedge^{y_m}\mathbf{Q}^n$$

is $p_n^*(\mathbf{y})$*, while the multiplicity of the sign representation of* S_n *is* $q^*(\mathbf{y})$.

3.16. Trivial and Sign Characters in Weyl Modules. Using the Jacobi-Trudi determinants § 3.5, it is possible to use the results of § 3.13 and § 3.15 to give combinatorial interpretations for the trivial and sign representations in Weyl modules:

THEOREM. *Let* λ *be an integer partition with* l *parts and largest part of size* m.

(1) *The multiplicity of the trivial representation of* S_n *in* $W_\lambda(\mathbf{Q}^n)$ *is given by*

$$\sum_{w\in S_n} \operatorname{sgn}(w) p_n(\lambda_1 - 1 + w(1), \ldots, \lambda_l - l + w(l))$$

and

$$\sum_{w\in S_n} \operatorname{sgn}(w) p_n^*(\lambda_1' - 1 + w(1), \ldots, \lambda_m - m + w(m)).$$

(2) *The multiplicity of the sign representation of* S_n *in* $W_\lambda(\mathbf{Q}^n)$ *is given by*

$$\sum_{w\in S_n} \operatorname{sgn}(w) q_n(\lambda_1 - 1 + w(1), \ldots, \lambda_l - l + w(l))$$

and

$$\sum_{w\in S_n} \operatorname{sgn}(w) q_n^*(\lambda_1' - 1 + w(1), \ldots, \lambda_m - m + w(m)).$$

EXAMPLE. Let $\lambda = (2,2,1)$. Note that for $n \geq |\mathbf{x}|$, $p_n(\mathbf{x})$ is constant. Thus, for $n \geq 5$, the multiplicity of the trivial representation of S_n in $W_\lambda(\mathbf{Q}^n)$ is given by

$$p_5(2,2,1) - p_5(2,3,0) - p_5(3,1,1) + p_5(4,1,0) = 26 - 16 - 21 + 12 = 0,$$

or by

$$q_5(3,2) - q_5(4,1) = 1 - 0 = 1.$$

For fixed $\mathbf{x}$, $q_n(\mathbf{x}) = 0$ for $n \geq |x| + 2$ (since, in any vector partition of $\mathbf{x}$ with more than $|\mathbf{x}| + 1$ parts, the zero vector must occur more than once). Hence the sign representation will not occur in $W_\lambda(\mathbf{Q}^n)$ for $n > |\lambda| + 1$.

EXAMPLE 3.3. In $W_{(3,2,2)}(\mathbf{Q}^3)$, the multiplicity of the sign representation of S_3 is

$$q_3^*(3,2) - q_3^*(4,1) = 0.$$

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