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$\lambda$ - integer partition of $n$. 

The problem
The map $w \mapsto \det(\rho_{\lambda}(w))$ is either the trivial character, or the sign character of $S_n$.
We call $\lambda$ chiral if $w \mapsto \det(\rho_{\lambda}(w))$ is the sign character of $S_n$.

For how many partitions of $n$ are chiral?

Definition $b(n) =$ number of chiral partitions of $n$. 

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The OEIS Foundation is grateful to everyone who made a donation during our Annual Appeal. Visit the new and spectacular Pictures from the OEIS page!

Search: a045923
Displaying 1-1 of 1 result found.
Sort: relevance | references | number | modified | created | Format: long | short | data

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OFFSET
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COMMENTS
Irreducible representations of S_n contained in the special linear group were first considered by L. Solomon (unpublished).

REFERENCES

LINKS
Table of n, a(n) for n=1..30.

EXAMPLE
a(5)=2, since only the irreducible representations indexed by the partitions (5) and (3,2) are contained in the special linear group.

KEYWORD
nonn,nice

AUTHOR
Richard Stanley

STATUS
approved
Closed Formula for number of representations of $S_n$ with non-trivial determinant

Suppose $n$ has binary expansion:

$$n = \epsilon + 2^{k_1} + 2^{k_2} + \cdots + 2^{k_r}, \text{ with } 0 < k_1 < \cdots < k_r, \epsilon \in \{0, 1\},$$
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Then the number of partitions $\lambda$ of $n$ for which $w \mapsto \det(\rho_{\lambda}(w))$ is the sign character is

$$2^{k_2 + \cdots + k_r} \left( 2^{k_1 - 1} + \sum_{\nu=1}^{k_1-1} 2^{(\nu+1)(k_1-2)-\binom{\nu}{2}} + \epsilon 2^{k_1-2} \right).$$

Example
Take $n = 41 = 1 + 2^{3} + 2^{5}$. So $\epsilon = 1$, $k_1 = 3$, and $k_2 = 5$.

$$b(41) = 2^{5} \times \left( 2^{3-1} + \sum_{\nu=1}^{3-1} 2^{(\nu+1)(3-2)-\binom{\nu}{2}} + 1 \times 2^{3-2} \right) = 640.$$
Closed Formula for number of representations of \( S_n \) with non-trivial determinant

Suppose \( n \) has binary expansion:

\[
n = \epsilon + 2^{k_1} + 2^{k_2} + \cdots + 2^{k_r}, \quad \text{with } 0 < k_1 < \cdots < k_r, \epsilon \in \{0, 1\},
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Then the number of partitions \( \lambda \) of \( n \) for which \( w \mapsto \det(\rho_\lambda(w)) \) is the sign character is

\[
2^{k_2 + \cdots + k_r} \left( 2^{k_1 - 1} + \sum_{v=1}^{k_1-1} 2^{v+1}(k_1-2) - \binom{v}{2} + \epsilon 2^{\binom{k_1}{2}} \right).
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Closed Formula for number of representations of $S_n$ with non-trivial determinant

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A Combinatorial Interpretation of the Determinant

The vector space $V_{\lambda}$ has a basis (Young's orthogonal form)

$\{ v_T | T \text{ a standard tableau of shape } \lambda \}$.

$\rho_{\lambda}(s_i)v_T = v_T$ if $i$ and $i+1$ are in the same row of $T$,

$-v_T$ if $i$ and $i+1$ are in the same column of $T$,

If neither case holds, then the action is more complicated.

But 1 and 2 are always in the same row or same column.

The vectors $v_T$ are eigenvectors of $\rho_{\lambda}(s_1)$ with eigenvalue $\pm 1$.

Let $g_{\lambda}$ denote the number of standard tableaux with 1 and 2 in the same column.

Conclusion

$\det \circ \rho_{\lambda}$ is the sign character if and only if $g_{\lambda}$ is odd.
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Relation to the character value at \((2, 1^{n-2})\)
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\[
f_{\lambda} \quad - \quad \text{number of SYT of shape } \lambda,
\]

\(f_{\lambda}\) - number of SYT of shape \(\lambda\),
Relation to the character value at \((2, 1^{n-2})\)

\[
\begin{array}{ll}
f_\lambda & \text{- number of SYT of shape } \lambda, \\
\dim(V_\lambda) & \\
\end{array}
\]

The character value \(\chi_\lambda(2, 1^{n-2})\) has a nice formula:

\[
\chi_\lambda(2, 1^{n-2}) = f_\lambda C(\lambda) \left( \binom{n-2}{2} \right).
\]

(Macdonald, *Symmetric functions and Hall polynomials*, p. 118, using the theory of skew-Schur functions)
Relation to the character value at \((2, 1^{n-2})\)

\[\begin{array}{ll}
  f_\lambda & - \text{number of SYT of shape } \lambda, \\
  \dim(V_\lambda) & \\
  g_\lambda & - \text{number of such SYT with 1 and 2 in the same column}
\end{array}\]

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\square & \square & \\
\square & \\
\end{array}
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\hline
& & \\
& & \\
& & \\
\hline
\end{array}
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\[
C(\lambda) = \sum \begin{array}{cccc}
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-1 & 0 & & \\
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\lambda = (4, 2) = \begin{array}{cccc}
\text{cell} & \text{cell} & \text{cell} & \text{cell} \\
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\end{array}
\]

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C(\lambda) = \sum \begin{array}{cccc}
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Character Formula:

\[
\chi_{\lambda}(2, 1^{n-2}) = \frac{f_{\lambda} C(\lambda)}{\binom{n}{2}}.
\]
Formula for $g_\lambda$

\[
g_\lambda = \left( f_\lambda - \chi_\lambda(2, 1^{n-2}) \right)/2
= f_\lambda \left( \frac{\left( \begin{array}{c} n \\ 2 \end{array} \right) - C(\lambda)}{\left( \begin{array}{c} n \\ 2 \end{array} \right)} \right)
\]
Formula for $g_\lambda$

$$g_\lambda = (f_\lambda - \chi_\lambda(2, 1^{n-2}))/2$$

$$= f_\lambda \left( \frac{n}{2} - C(\lambda) \right)$$

So $\rho_\lambda$ is chiral if and only if:

$$\nu_2(f_\lambda) + \nu_2\left( \left( \frac{n}{2} \right) - C(\lambda) \right) = \nu_2\left( \frac{n}{2} \right)$$
To understand $v_2(f_{\lambda})$
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One ingredient is the hook-length formula (Frame, Robinson and Thrall):

$$f_\lambda = \frac{n!}{\prod_{(i,j) \in \lambda} h(i,j)}$$

Example

Hook-lengths of $(4,2)$ are

$$5 \ 4 \ 2 \ 1 \ 2 \ 1$$

so

$$f(4,2) = 6! \times 5 \times 4 \times 2 \times 1 \times 2 \times 1 = 9.$$
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$$\lambda \leftrightarrow (\text{core}_p \lambda, \text{quo}_p \lambda).$$
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The partition $\text{core}_p \lambda$ is what remains of Young diagram of $\lambda$ after successively removing the rims of as many $p$-hooks as possible. The $p$-quotient $\text{quo}_p \lambda$ is a $p$-tuple $(\lambda_0, \ldots, \lambda_{p-1})$ of partitions.
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$$|\lambda| = |\text{core}_p \lambda| + p(|\lambda_0| + \cdots + |\lambda_{p-1}|).$$

The size of the partition $\lambda_k$ in the $p$-quotient is the number of nodes in the Young diagram of $\lambda$ whose hook-lengths are multiples of $p$, and whose hand-nodes have content congruent to $k$ modulo $p$ (by definition, the content of the node $(i,j)$ is $j - i$). The partition $\lambda$ can be recovered uniquely from $\text{core}_p \lambda$ and $\text{quo}_p \lambda$. 
Example of core

The 2-core of (5, 4, 2, 2, 1) is (3, 2, 1):
Example of quotient

The hook-lengths of \((5, 4, 2, 2, 1, 1)\) are:

\[
\begin{array}{cccc}
10 & 7 & 4 & 3 \\
8 & 5 & 2 & 1 \\
5 & 2 & & \\
4 & 1 & & \\
2 & & & \\
1 & & & \\
\end{array}
\]
Example of quotient

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And its 2-quotient is given by
A result from Frame-Robinson-Thrall

Lemma
There exists a bijection from the set of cells in $\text{quo}_p \lambda$ onto the set of cells in $\lambda$ whose hook-lengths are divisible by $p$ under which a cell of hook-length $h$ in $\text{quo}_p \lambda$ is mapped to a cell of hook-length $ph$ in $\lambda$. 
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Recursive Criterion for odd Dimensionality

If $n$ has binary expansion

$$n = \epsilon + 2^{k_1} + 2^{k_2} + \cdots + 2^{k_r}, \text{ with } 0 < k_1 < \cdots < k_r, \epsilon \in \{0, 1\},$$

and $\lambda$ is a partition of $n$ with $\text{core}_2\lambda$ of size $a$, and $\text{quo}_2\lambda$ having partitions $\mu_0$ and $\mu_1$ of sizes $m_0$ and $m_1$ (so $n = a + 2m_0 + 2m_1$),

Theorem (Macdonald)

$f_\lambda$ is odd if and only if

- $\epsilon = a$, 
- The binomial coefficient $\frac{n - \epsilon \cdot (2^{m_0} \cdot 2^{m_1})}{(n - \epsilon)!}$ is odd,
- $f_\mu_0$ and $f_\mu_1$ are odd.

Remark

The binomial coefficient $\binom{n}{k}$ is odd if and only if the binary digits of $k$ and $n - k$ are in disjoint positions.
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Recursion of core and quotient construction: the core tower

If $\lambda$ is a partition with core $\alpha$ and quotient $\mu_0$ and $\mu_1$, then its 2-core tower $T(\lambda)$ is a binary tree, defined recursively as follows:
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If $\lambda$ is a partition with core $\alpha$ and quotient $\mu_0$ and $\mu_1$, then its 2-core tower $T(\lambda)$ is a binary tree, defined recursively as follows:

$$T(\lambda) = \begin{cases} \text{core}_2 \lambda & \text{if } \lambda \text{ is a 2-core partition} \\ T(\mu_0) & \text{if } \mu_0 \text{ is the first quotient} \\ T(\mu_1) & \text{if } \mu_1 \text{ is the second quotient} \end{cases}$$
Example of 2-core tower

The 2-core tower of \((5, 4, 2, 2, 1, 1)\) is:

\[
\emptyset \emptyset \emptyset \emptyset \emptyset \emptyset \emptyset
\]

Let \(w_i(\lambda) = \) sum of sizes of entries in \(i\)th row. Here:

\(w_i(\lambda) = 1\) for \(i = 0, 1, 2, 3\), and 0 otherwise.
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```
∅          (1)
∅          ∅
∅          ∅
∅          ∅
∅      (1)  ∅
∅      ∅    ∅
∅      ∅    ∅
∅    ∅    ∅    ∅    ∅    ∅    ∅    ∅
```

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Olsson’s criterion for \( f_\lambda \) being odd

Let \( \nu_i(n) \) be the \( i \)th digit in the binary expansion of \( n \).

So \( n = \sum_i 2^{\nu_i(n)} \).

Theorem

Let \( \lambda \) be a partition of \( n \). Then \( f_\lambda \) is odd if and only if

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The 2-core tower of $\lambda = (5, 4, 2, 2, 1, 1)$ (a partition of 15) is:

$$
\emptyset \emptyset \emptyset \emptyset \emptyset \emptyset \emptyset 
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$f(\lambda)$ is odd.
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\[ f_\lambda \text{ is odd.} \]
Counting odd dimensional representations

Theorem (Macdonald, bijective proof via Olsson)

The number of partitions $\lambda$ of $n$ such that $f_\lambda$ is odd is

$$2 \sum_i \nu_i(n).$$

To prove, count the possible 2-core towers.
Theorem (Macdonald, bijective proof via Olsson)
The number of partitions $\lambda$ of $n$ such that $f_\lambda$ is odd is $2\sum_i i\nu_i(n)$. 
Counting odd dimensional representations

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In the example of $(5, 4, 2, 2, 1, 1)$:
Contents in a rim-hook

\[
C(h_3) = 2 \times 2^3 - \binom{2^3}{2}
\]
The head node contribution is even on the left side of the tree and odd on the right side.
2-core towers of chiral partitions

Chiral partitions $\lambda$ of $n$ are partitions for which $v_2(f_{\lambda}) + v_2((n^2 - C(\lambda)) = v_2((n^2)$.

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2-core towers of chiral partitions

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By carefully keeping track of the contributions of different rim-hooks, we were able to characterize the 2-core towers of chiral partitions.
If \( n = \epsilon + 2^{k_1} + 2^{k_2} + \cdots + 2^{k_r} \), with \( 0 < k_1 < \cdots < k_r, \epsilon \in \{0, 1\} \). Then a partition \( \lambda \) of \( n \) is chiral if and only if one of the following happens:

1. The partition \( \lambda \) satisfies
   \[
   w_i(\lambda) = \begin{cases} 
   1 & \text{if } i \in \{k_1, \ldots, k_r\}, \text{ or if } \epsilon = 1 \text{ and } i = 0, \\
   0 & \text{otherwise,}
   \end{cases}
   \]
   and the unique non-trivial partition in the \( k_1 \)th row of the 2-core tower of \( \lambda \) is \( \alpha_x \), where the binary sequence \( x \) of length \( k \) begins with \( \epsilon \). In this case \( f_{\lambda} \) is odd.

2. For some \( 0 < v < k_1 \),
   \[
   w_i(\lambda) = \begin{cases} 
   2 & \text{if } i = k_1 - v, \\
   1 & \text{if } k_1 - v + 1 \leq i \leq k_1 - 1 \text{ or } i \in \{k_2, \ldots, k_r\}, \\
   & \text{or if } \epsilon = 1 \text{ and } i = 0, \\
   0 & \text{otherwise,}
   \end{cases}
   \]
   and the two non-trivial partitions in the \( (k - v) \)th row of the 2-core tower of \( \lambda \) are \( \alpha_x \) and \( \alpha_y \), for binary sequences \( x \) and \( y \) such that \( x \) begins with \( 0 \) and \( y \) begins with \( 1 \). In this case \( v_2(f_{\lambda}) = v \).

3. We have \( \epsilon = 1 \) and the partition \( \lambda \) satisfies
   \[
   w_i(\lambda) = \begin{cases} 
   3 & \text{if } i = 0, \\
   1 & \text{if } i \in \{1, \ldots, k_1 - 1, k_2, \ldots, k_r\}.
   \end{cases}
   \]
   In this case, \( v_2(f_{\lambda}) = k_1 \).
Counting such towers gives:

If \( n = \epsilon + 2^{k_1} + 2^{k_2} + \cdots + 2^{k_r} \), with \( 0 < k_1 < \cdots < k_r, \epsilon \in \{0, 1\} \), then the number \( b_\nu(n) \) of chiral partitions \( \lambda \) of \( n \) for which \( \nu_2(f_\lambda) = \nu \) is given by

\[
b_\nu(n) = 2^{k_2+\cdots+k_r} \times \begin{cases} 
2^{k_1-1} & \text{if } \nu = 0, \\
2^{(\nu+1)(k_1-2)-\binom{\nu}{2}} & \text{if } 0 < \nu < k_1, \\
\epsilon 2^{\binom{k_1}{2}} & \text{if } \nu = k, \\
0 & \text{if } \nu > k_1.
\end{cases}
\]
Counting such towers gives:

If \( n = \epsilon + 2^{k_1} + 2^{k_2} + \cdots + 2^{k_r} \), with \( 0 < k_1 < \cdots < k_r \), \( \epsilon \in \{0, 1\} \), then the number \( b_\nu(n) \) of chiral partitions \( \lambda \) of \( n \) for which \( \nu_2(f_\lambda) = \nu \) is given by

\[
b_\nu(n) = 2^{k_2 + \cdots + k_r} \times \left\{ \begin{array}{ll}
2^{k_1-1} & \text{if } \nu = 0, \\
2(\nu+1)(k_1-2)-\binom{\nu}{2} & \text{if } 0 < \nu < k_1, \\
\epsilon 2^{k_1-\nu} & \text{if } \nu = k, \\
0 & \text{if } \nu > k_1.
\end{array} \right.
\]

\[
b(n) = 2^{k_2 + \cdots + k_r} \left( 2^{k_1-1} + \sum_{\nu=1}^{k_1-1} 2(\nu+1)(k_1-2)-\binom{\nu}{2} + \epsilon 2^{k_1-\nu} \right).
\]
Let $a(n)$ be the number of partitions of $n$ for which $f_\lambda$ is odd. Recall $b(n)$ is the number of chiral partitions of $n$. 
Growth

Let $a(n)$ be the number of partitions of $n$ for which $f_\lambda$ is odd. Recall $b(n)$ is the number of chiral partitions of $n$.

$$a(n) = 2^{k_1 + \cdots + k_r},$$

$$b(n) = 2^{k_2 + \cdots + k_r} \left( 2^{k_1 - 1} + \sum_{v=1}^{k_1 - 1} 2(v+1)(k_1 - 2) - \binom{v}{2} + \epsilon_2 \binom{k_1}{2} \right).$$
Comparison of $a(n)$ and $b(n + 2)$

$\frac{2}{5} \leq \frac{a(n)}{b(n + 2)} \leq 1.$
Growth

For all $n$: 
\[ n \leq a(n) \leq 2^{\log_2(n+1)(\log_2(n+1) - 1)/2} \]
Growth

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Hardy-Ramanujan formula:

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp(\pi \sqrt{2n/3}) \text{ as } n \to \infty.$$
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Hardy-Ramanujan formula:

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So $a(n)/p(n) \to 0$, and $b(n)/p(n) \to 0$. 
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Representations of symmetric groups with non-trivial determinant

Arvind Ayyer, Amitanshu Prasad, Steven Spallone

(Submitted on 29 Apr 2016)

We give a closed formula for the number of partitions $\lambda$ of $n$ such that the corresponding irreducible representation $S_n^\lambda$ of $S_n$ has non-trivial determinant. We determine how many of these partitions are self-conjugate and how many are hooks. This is achieved by characterizing the $2$-core towers of such partitions. We also obtain a formula for the number of partitions of $n$ such that the associated permutation representation of $S_n$ has non-trivial determinant.

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