$$G_k(\infty) = 2\zeta(2k).$$

$$\frac{\cancel{z}\, z}{e^z - 1} = \sum_{k=0}^{\infty} \frac{B_k z^k}{k!}$$

$B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_{2k+1} = 0$ for $k > 0$

$$B_2 = \frac{1}{6},\quad B_4 = -\frac{1}{30},\quad B_6 = \frac{1}{42}.$$

$$\zeta(2k) = (-1)^{k-1}\, 2^{2k-1}\, \pi^{2k}\, \frac{B_{2k}}{(2k)!}$$

$$\zeta(4) = \frac{\pi^4}{90},\quad \zeta(6) = \frac{\pi^6}{7\times 3^3\times 5}$$

$$G_2(\infty) = \frac{\pi^4}{45},\quad G_3(\infty) = \frac{2\cancel{8}\pi^6}{\cancel{27}\; 7\times 3^3\times 5}.$$

$$g_2 = 60G_2 \qquad g_3 = 140G_3$$

$$g_2(\infty) = \frac{4}{3}\pi^4,\quad g_3(\infty) = \frac{8\pi^6}{27}.$$

$$\boxed{\Delta} = g_2^3 - 27g_3^2$$

$$\Delta(\infty) = \left(\frac{64}{27} - \frac{64}{27}\right)\pi^{12} = 0.$$

So Δ is a cusp form of wt. 12

If f is a modular form of wt $2k$

$$f \in M_k$$

$$(*)\qquad v_\infty(f) + \frac{1}{2}v_i(f) + \frac{1}{3}v_\rho(f) + \sum_{\substack{p\in PSL_2(\mathbb{Z})\backslash H \\ p\notin PSL_2(\mathbb{Z})i,\ PSL_2(\mathbb{Z})\rho}} v_p(f) = \frac{k}{6}.$$

---

1. $M_k = 0$ for $k < 0$ & $k = 1$
2. M_i is spanned by G_i for $i = 2, \dots, 5$.
3. $M_{k-6} \xrightarrow{\times\Delta} M_k^0$ is an iso.
4. $\dim M_k = \begin{cases} \lfloor \frac{k}{6} \rfloor & \text{if } k \equiv 1 \bmod 6 \\ \lfloor \frac{k}{6} \rfloor + 1 & \text{o/w}. \end{cases}$

$$j = 1728\, g_2^3 / \Delta$$

j is a modular fn. of wt. 0

→ j is hol. on H

→ j has a simple pole at ∞.

$$j : {}^{PSL_2(\mathbb{Z})} \backslash H \longrightarrow \mathbb{C}.$$

($PSL_2(\mathbb{Z})$ " G)

Thm: j is a bijection.

Pf: Given any $\lambda \in \mathbb{C}$, T.P.T.

$\exists !$ $z \in {}^{G}\backslash H$ such that

$$\frac{1728\, g_2^3(z)}{\Delta(z)} = \lambda$$

$$1728\, g_2^3(z) - \lambda \Delta(z) = 0$$

$\Leftrightarrow f(z) = 1728\, g_2^3(z)$

Applying $(*)$ to $f(z)$ gives:

$$\underbrace{\tfrac{1}{2} v_i(f)}_{a} + \underbrace{\tfrac{1}{3} v_\rho(f)}_{b} + \underbrace{\sideset{}{^*}\sum_{\substack{p \in G\backslash H \\ p \notin Gi,\, G\rho}} v_p(f)}_{c} = 1$$

$$\tfrac{1}{2} a + \tfrac{1}{3} b + c = 1$$

The only solns. are

$a = 2,\ b = 0,\ c = 0$

$a = 0,\ b = 3,\ c = 0$

$a = 0,\ b = 0,\ c = 1.$

so f has a unique root.

Thm: The following are equivalent

(i) If f is a modular fn. of wt. 0.

(ii) f is a quotient of 2 modular forms of the same weight.

(iii) f is a rational fn. of j.

Pf. Clearly (iii) ⇒ (ii) ⇒ (i)

T.P.T (i) ⇒ (iii).

Suppose f is a modular fn of wt. 0.

If f has a pole at $p \in H$,

$f(z)[j(z) - j(p)]^k$ will not .

so, after multiplying f by a rational fn. of j, can assume that f has no poles in H.

f may have a pole ~~at~~ at ∞.

$f\Delta^n$ ~~is~~ has no pole at ∞ (is a mod form)

~~Recall~~ Claim:
M_k is spanned by
$$\{ G_2^\alpha G_3^\beta \mid 2\alpha+3\beta = k\}$$
$f \in M_k$.

Use induction: Choose any α, β such that $2\alpha+3\beta = k$.

$f - \lambda G_2^\alpha G_3^\beta$ is a cusp form for some λ

$= \Delta h$, $h \in M_{k-6}$.

~~so~~ So now use induction.

$f\Delta^n$ lies in the span of $G_2^\alpha G_3^\beta$.

f lies in the span of

$$\frac{G_2^\alpha G_3^\beta}{\Delta^n} \qquad 2\alpha+3\beta = 6n.$$

$$= \left(\frac{G_2^3}{\Delta}\right)^a \left(\frac{G_3^2}{\Delta}\right)^b$$

$$2\alpha+3\beta = 6n \qquad a = \frac{\alpha}{3},\ b = \frac{\beta}{2} \text{ are integers.}$$

so, it suffices to show that

G_2^3/Δ & G_3^2/Δ

are rational functions of j.

$$G\backslash H \xrightarrow{j} \mathbb{C} \subseteq \hat{\mathbb{C}}$$

The partial fraction expansion of

$$\pi \cot \pi z = \frac{\pi \cos \pi z}{\sin \pi z}$$

has simple poles at $m \in \mathbb{Z}$.

The pricipal part at $z = m$ is

$$\frac{1}{z-m}.$$

"$$\pi \cot \pi z = \sum_{m=-\infty}^{\infty} \frac{1}{z-m} + \text{entire}$$"

Thm: $$\pi \cot \pi z = \underbrace{\frac{1}{z} + \sum_{m=1}^{\infty} \left[\frac{1}{z+m} + \frac{1}{z-m}\right]}_{f(z).}$$

Doubling formula:

$$2\pi \cot 2\pi z = \pi \cot \pi z + \pi \cot \left[\pi\left(z+\tfrac{1}{2}\right)\right]$$

(†) $$f_n(z) = \frac{1}{z} + \sum_{m=1}^{n} \frac{1}{z+m} + \frac{1}{z-m}.$$

Claim: $2f(2z) = f(z) + f(z+\tfrac{1}{2})$.

$$f_n(z) + f_n(z+\tfrac{1}{2})$$

$$= \frac{1}{z+\frac{1}{2}} + \frac{1}{z} + \sum_{m=1}^{n} \frac{1}{z+m} + \frac{1}{z+m+\frac{1}{2}} + \frac{1}{z-m} + \frac{1}{z-m+\frac{1}{2}}$$

$$= \frac{2}{2z} + \frac{2}{2z+1} + \sum_{m=1}^{n} \frac{2}{2z+2m} + \frac{2}{2z+2m+1} + \frac{2}{2z-2m} + \frac{2}{2z-2m+1}$$

$$= \cancel{2 f_{2n}(2z)}\ 2 f_{2n}(2z) + \text{suitly small in } n$$

Takiy $\lim_{n\to\infty}$ gives (†).

Now take:

$$g(z) = 2\pi\cot 2\pi z - f(z)$$

$g(0) = 0$, g is entire.

By the maximum modulus principle, if $g \neq 0$.

$\exists\ |c| = 2$, $\ni$

$$|g(c)| > |g(z)| \quad \forall z \in B_2(0).$$

$$2g(c) = g\left(\frac{c}{2}\right) + g\left(\frac{c+1}{2}\right)$$

(by doubling fmla)

$$|2g(c)| \leq \left|g\left(\frac{c}{2}\right)\right| + \left|g\left(\frac{c+1}{2}\right)\right|$$

$$< |g(c)| + |g(c)| = 2|g(c)|$$

a contradiction.

Ex **Power-series expansion of Eisenstein series.**

$$G_k(z) = \sum_{(m,n)\neq(0,0)} \frac{1}{(nz+m)^{2k}}.$$

$$= 2\zeta(2k) + 2\sum_{n=1}^{\infty}\sum_{m\in\mathbb{Z}} \frac{1}{(nz+m)^{2k}}$$

$$\cos z = \frac{e^{iz}+e^{-iz}}{2}$$

$$\sin z = \frac{e^{iz}-e^{-iz}}{2i}$$

$$\pi\cot\pi z = \pi i\,\frac{e^{\pi i z}+e^{-\pi i z}}{e^{\pi i z}-e^{-\pi i z}}$$

$$= \pi i\,\frac{q+1}{q-1} = \pi i - \frac{2\pi i}{1-q}$$

$$= \pi i - 2\pi i\sum_{n=0}^{\infty} q^n.$$

Have:

$$\frac{1}{z} + \sum_{m=1}^{\infty}\left[\frac{1}{z+m} + \frac{1}{z-m}\right] = \pi i - 2\pi i\sum_{n=0}^{\infty} q^n$$

Differentiating both sides $\cancel{(2k-1)}$ $k-1$ times

$$\sum_{m\in\mathbb{Z}} \frac{1}{(m+z)^{k}} = \frac{1}{(k-1)!}(-2\pi i)^{k}\sum_{a=1}^{\infty} a^{k-1} q^{a}.$$

So $G_k(z)$

$$= 2\zeta(2k) + 2\sum_{n=1}^{\infty}\sum_{m\in\mathbb{Z}} \frac{1}{(nz+m)^{2k}}$$

$$= 2\zeta(2k) + 2\sum_{n=1}^{\infty}\sum_{a=1}^{\infty} \frac{(2\pi i)^{2k}}{(2k-1)!} a^{2k-1} q^{an}$$

$$= 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!}\sum_{m=1}^{\infty}\left(\sum_{an=m} a^{2k-1}\right) q^{m}$$

Define: $\sigma_k(n) = \sum_{d/n} d^{k}$

$$= 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!}\sum_{m=1}^{\infty} \sigma_{2k-1}(m)\, q^{m}$$

Arithmetic fns:

$f : \mathbb{N} \to \mathbb{C}$

$$f * g(n) = \sum_{d|n} f(d)\, g\left(\frac{n}{d}\right).$$

$\mathbb{N}$ is a partially ordered set

$d \leq n$ if $d|n$.

$P_1, P_2, \ldots$ partially order sets

$\prod_{i=1}^{\infty} P_i$ has a partial order.

$(\alpha_1, \alpha_2, \ldots) \leq (\beta_1, \beta_2, \ldots)$

$\Leftrightarrow \alpha_1 \leq \beta_1, \alpha_2 \leq \beta_2, \ldots$

$f(mn) = f(m) f(n)$ when $(m,n) = 1$

$\to$ Möbius inversion ...