(1)

$\mathbb{H}$

$f : \mathbb{H} \longrightarrow \hat{\mathbb{C}}$

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^{2k} f(z)$$

<u>weight $2k$</u>

(also periodic w/ period 1)

$$f(z+1) = f(z)$$

$\exists \quad \tilde{f} : B_1(0)^* \longrightarrow \hat{\mathbb{C}}$ such that

$$f(z) = \tilde{f}(e^{2\pi i z}) = \tilde{f}(q).$$

<u>Modular form:</u> $\tilde{f}$ extends to a hol. fn. on $B_1(0)$

$$z = \infty \longleftrightarrow q = 0$$

<u>Cusp form:</u> $f(\infty) := \tilde{f}(0) = 0$.

(2)

$$G_k(z) = \sum_{(m,n)\neq 0} \frac{1}{(mz+n)^{2k}}$$

$$= \sum_{\gamma \in \Gamma(z,1) - \{0\}} \frac{1}{\gamma^{2k}}.$$

Converges abs. for $k > 1$

$$G_k(\infty) = 2\zeta(2k)$$

$$g_2 := 60\,G_2, \quad g_3 = 140\,G_3$$

Then $\quad \Delta = g_2^3 - 27 g_3^2$

Δ is a cusp form of weight 12.

see "Modular Functions & Dirichlet series by Tom M. Apostol." for relu. to elliptic curves.

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$v = v_p(f)$ $\quad \boxed{f(z) = (z-p)^v \tilde{f}(z)}$

$\tilde{f}(p) \neq 0$ hol at p.

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \sum_p \mathrm{ind}_\gamma(p) v_p(f)$$

<u>Thm:</u> Let f be a modular <u>function</u> of weight $2k$, $f \not\equiv 0$. Then

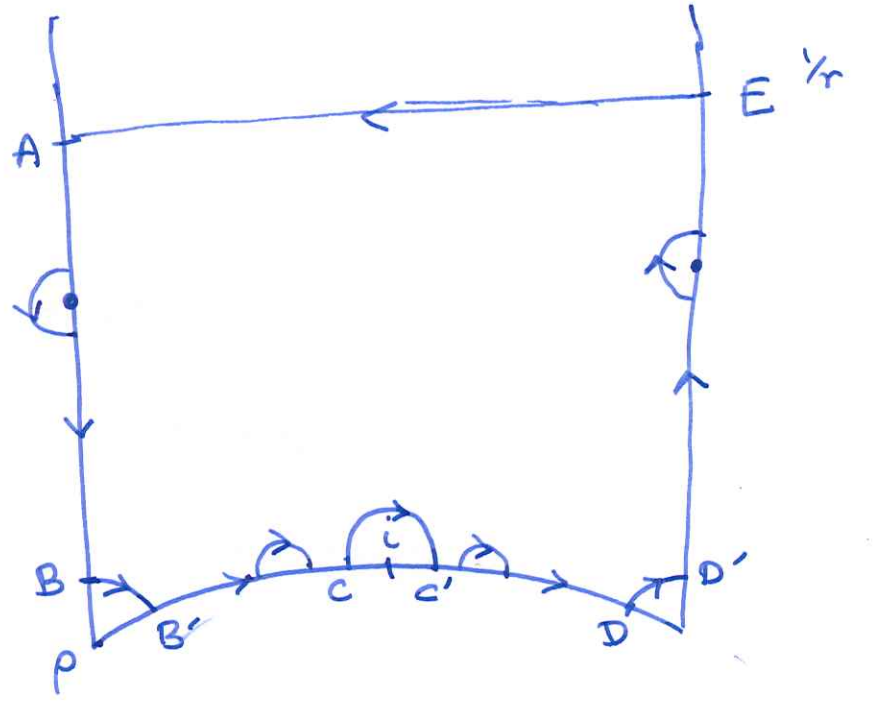
$$v_\infty(f) + \frac{1}{2} v_i(f) + \frac{1}{3} v_\rho(f)$$

$G = PSL_2(\mathbb{Z})$ $\quad v_0(f)$

$$+ \sum_{\substack{p \in G \backslash H \\ p \notin \{Gi, G\rho\}}} v_p(f) = \frac{k}{6}.$$

<u>Pf:</u> ~~Fix r > 0, and assu~~

Assume f has no 0's or poles on ∂D, except maybe at i, ρ and $-\bar{\rho}$.



Let γ be the contour in the figure (depends on r)

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \sum_{\substack{p \in G \backslash H \\ p \notin \{Gi, G\rho\}}} v_p(f)$$

(T) 1 $\displaystyle \frac{1}{2\pi i} \int_A^B \frac{f'(z)}{f(z)} dz + \frac{1}{2\pi i} \int_{D'}^E \frac{f'(z)}{f(z)} dz = 0$

(S) 2 S maps $B'C$ to DC'

$$f(Sz) = z^{2k} f(z)$$

$$d f(Sz) = \left[2k z^{2k-1} f(z) + z^{2k} f'(z)\right] dz$$

$$\frac{df(Sz)}{f(Sz)} = 2k \cdot \frac{dz}{z} + \frac{f'(z)}{f(z)}\, dz$$

$$\frac{1}{2\pi i}\left[\int_{B'}^{C} \frac{f'(z)}{f(z)}\, dz + \int_{C'}^{D} \frac{f'(z)}{f(z)}\, dz\right]$$

$$\frac{1}{2\pi i}\left[\int_{B'}^{C} \left(\cancel{\frac{f'(z)}{f(z)}} - \frac{2k}{z} - \cancel{\frac{f'(z)}{f(z)}}\right) dz\right]$$

$$\xrightarrow[r\to 0]{} \quad -2k\left(-\frac{k}{12}\right) = \frac{k}{6}.$$

Get: (2)

$$\frac{1}{2\pi i}\int_{A}^{B} \frac{f'(z)}{f(z)}\, dz + \frac{1}{2\pi i}\int_{D'}^{E'} \frac{f'(z)}{f(z)}\, dz$$

$$= \frac{k}{6}.$$

(3) $$\int_{E}^{A} \frac{f'(z)}{dz}\, dz = -v_\infty(f)$$

(4) $$\int_{B^*}^{B'} \frac{f'(z)}{f(z)}\, dz \to -\frac{1}{6} v_\rho(f)$$

$$f(z) = (z-p)^{v}\, \bar{f}(z) \qquad \bar{f}(p) \neq 0$$

$\tilde{f}$ hol. at p.

$$\frac{f'(z)}{f(z)} = \frac{v}{z-p} + \text{hol. fn.}$$

(5) $$\int_{C}^{C'} \frac{f'(z)}{f(z)}\, dz \to -\frac{1}{2} v_i(f)$$

(6) $$\int_{D}^{D'} \frac{f'(z)}{f(z)}\, dz = -\frac{1}{6} v_\rho(f)$$

Get:

$$\sum_{\substack{p \in G\backslash H \\ p \notin \{Gi, G\rho\}}} v_p(f) = -v_\infty(f) + \frac{k}{6} - \frac{1}{3} v_\rho(f) - \frac{1}{2} v_i(f)$$

(*) $v_\infty(f) + \frac{1}{3} v_\rho(f) + \frac{1}{2} v_i(f) + \sum' v_p(f) = \frac{k}{6}$

~~Thm:~~

M_k = space of modular forms of wt $2k$.

$M_k^0 \subseteq M_k$ cusp forms.

$$M_k^0 = \ker \begin{bmatrix} M_k \to \mathbb{C} \\ f \mapsto f(\infty) \end{bmatrix}$$

$$\dim M_k / M_k^0 \leq 1$$

$$\dim M_k / M_k^0 = 1 \text{ for } k > 1$$

$$M_k = \langle G_k \rangle + M_k^0 .$$

Thm: (i) $M_k = 0$ for $k<0$ and $k=1$.

(ii) M_0 is spanned by a const fn G_0

(iii) M_2, M_3, M_4, M_5 are spanned by G_2, G_3, G_4, G_5.

(iv) Mult. by Δ defines an iso.

$$M_{k-6} \to M_k^0 \quad \forall k.$$

Applying (*) to Δ,

$v_\infty(\Delta) = 1$, $v_p(\Delta) = 0 \; \forall p \in H$.

provided $\underline{\underline{\Delta \not\equiv 0}}$ ok.

Apply (*) to G_2

so $v_\rho(G_2) = 1$

& $v_p(G_2) = 0 \; \forall p \neq \rho$

$v_i(G_3) = 1$

$v_p(G_3) = 0 \; \forall p \neq i$.

Δ has a simple root at ∞ and no other zeroes/poles.

So $f \mapsto f/\Delta$ is a lin. map

$$M_k^0 \to M_{k-6}.$$

$$\dim M_k = \begin{cases} \lfloor k/6 \rfloor + 1 & \text{if } k \not\equiv 1 \ (6) \\ \lfloor k/6 \rfloor & \text{o/w}. \end{cases}$$