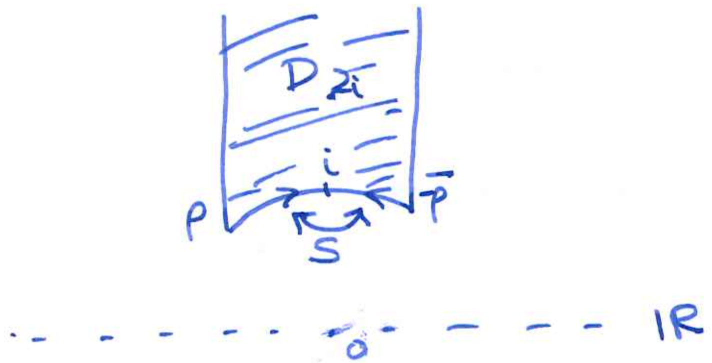
$H = \{ z \in \mathbb{C} \mid \cancel{\mathrm{Re}}\ \mathrm{Im}\, z > 0 \}$

$\mathrm{Aut}(H) = PSL_2(\mathbb{R}) \supset PSL_2(\mathbb{Z})$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

$D = \{ z \in H \mid \mathrm{Re}\, z \in [-\tfrac{1}{2}, \tfrac{1}{2}],\ |z| \geq 1 \}$



$$S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\mathrm{Stab}_{PSL_2(\mathbb{Z})} z = \begin{cases} \langle S \rangle & \text{if } z = i \\ \langle ST \rangle & \text{if } z = \rho \\ \langle TS \rangle & \text{if } z = -\bar{\rho} \\ \mathrm{id} & \text{o/w} \end{cases}$$

$PSL_2(\mathbb{Z}) = \langle S, T \rangle$

Elliptic curve

$E = \mathbb{C}/\Gamma$, where $\Gamma \subseteq \mathbb{C}$ is a lattice.

A lattice is a subgroup of $\mathbb{C}$ of the form $\langle \omega_1, \omega_2 \rangle$, where $\omega_1 \in \mathbb{C}$ and $\omega_2 \in \mathbb{C}$ are linearly indep. over $\mathbb{R}$.

$$\mathbb{C}/\Gamma \xrightarrow{\text{homeo}} \mathbb{R}^2/\mathbb{Z}^2 = T^2$$

$\exists\ g \in GL_2(\mathbb{R})$ such that

$$g e_1 = \omega_1$$
$$g e_2 = \omega_2$$

$$\mathbb{R}^2 \xrightarrow{g} \mathbb{C} \quad \Rightarrow \quad \mathbb{R}^2/\mathbb{Z}^2 \xrightarrow[\cong]{g} \mathbb{C}/\Gamma$$

$$\mathbb{Z}^2 \xrightarrow{g} \Gamma$$

(3) $E_1 = \mathbb{C}/\Gamma_1$ is iso to $E_2 = \mathbb{C}/\Gamma_2$

if $\exists$ biholomorphic map $\varphi : \mathbb{C} \to \mathbb{C}$

such that $\varphi(\Gamma_1) = \Gamma_2$.

So E_1 is iso $E_2 \iff \Gamma_1 = \lambda \Gamma_2$.

Γ_1 is homothetic to Γ_2.

$M := \{(\omega_1, \omega_2) \in \mathbb{C}^2 \mid \underbrace{\mathrm{Im}(\omega_1/\omega_2)}_{z \in H} > 0\}$

$\langle \omega_1, \omega_2 \rangle =: \Gamma(\omega_1, \omega_2)$

<u>Lemma</u>: $\Gamma(\omega_1, \omega_2) = \Gamma(\omega_1', \omega_2')$

$\iff \exists\ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$

such that $\omega_1' = a\omega_1 + b\omega_2$

$\omega_2' = c\omega_1 + d\omega_2$

$$z' = \frac{\omega_1'}{\omega_2'} = \frac{a\omega_1 + b\omega_2}{c\omega_1 + d\omega_2} = \frac{az+b}{cz+d}$$

Let $\mathcal{R}$ = set of all lattices in $\mathbb{C}$.

$$M \longrightarrow \mathcal{R}$$
$$(\omega_1, \omega_2) \longmapsto \Gamma(\omega_1, \omega_2)$$

Get a bijection:

$$SL_2(\mathbb{Z}) \backslash M \xrightarrow{\cong} \mathcal{R}$$

$\mathbb{C}^*$ acts on $SL_2(\mathbb{Z}) \backslash M \cong \mathcal{R}$

$$M/\mathbb{C}^* \xrightarrow{\cong} H$$
$$(\omega_1, \omega_2) \longmapsto \frac{\omega_1}{\omega_2}.$$

$\therefore\ SL_2(\mathbb{Z}) \backslash H \xrightarrow[\text{bij.}]{\cong} \mathbb{C}^* \backslash \mathcal{R} \longleftrightarrow$ iso. classes of elliptic curves.

($z \longmapsto \Gamma(z, 1)$

parametrizes elliptic curve "moduli space".

Defn: A meromorphic fn. $f : H \to \hat{\mathbb{C}}$ is said to be weakly modular of weight $2k$ if

$$(*) \quad f\left(\frac{az+b}{cz+d}\right) = (cz+d)^{2k} f(z).$$

$\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathbb{Z})$.
$\forall z \in H$.

Note: $PSL_2(\mathbb{Z}) = \langle S, T \rangle$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad g' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

$$gg' = \begin{pmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{pmatrix}$$

$$f(gg' \cdot z) = [(ca'+dc')z + (cb'+dd')]^{2k} f(z)$$

$\Updownarrow$?

$*$ holds for gg'

$*$ hold for g and for g'

$$f(gg', z) = \cancel{(cz+d)^2}\left(c\left(\frac{a'z+b}{c'z+d'}\right)+d\right)^{2k}(c'z+d')^{2k} f(z)$$

So $(*)$ holds for $\cancel{gg'}$ g & g'

$\Rightarrow$ $(*)$ holds for gg'.

$\therefore$ $(*)$ holds for all $g \in PSL_2(\mathbb{Z})$

$\Leftrightarrow$ $(*)$ holds for S & T.

$\Leftrightarrow$ $f\left(-\frac{1}{z}\right) = z^{2k} f(z) \quad \forall z \in H$

and

$$\boxed{f(z+1) = f(z)} \quad \forall z \in H$$

$\uparrow$ periodicity.

Periodic fns: $f : \mathbb{C} \to \mathbb{C}$ hol. is *periodic* (w/ period 1) if $f(z+1) = f(z)$.

$$z \longmapsto e^{2\pi i z}$$

$$\begin{tikzcd}
\mathbb{C} \arrow[r] \arrow[d] & \mathbb{C}^* \\
\mathbb{C}/\mathbb{Z} \arrow[ur, dashed, "\cong"] & \cup|
\end{tikzcd}$$

$$\cup|$$

$$H/\mathbb{Z} \xrightarrow{\cong} B_1(0)^*$$

$$\boxed{\lim_{\operatorname{Im} z \to +\infty} e^{2\pi i z} = 0}$$

$q = e^{2\pi i z}$

periodic.

<u>Upshot</u>: $f : H \to \mathbb{C}$ is a meromorphic fn., then $\exists\ \tilde{f} : B_1(0)^* \to \mathbb{C}$ such that $f(z) = \tilde{f}(q)$.

$$\tilde{f}(q) = \sum_{n=-\infty}^{\infty} a_n q^n = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n z}$$

(Laurent series exp)

f has a pole at ∞

$\iff \tilde{f}$ has pole at 0

f is hol. at ∞

$\iff \tilde{f}$ has removable sing at 0.

f vanishes at ∞

$\iff \tilde{f}$ vanishes at 0.

<u>Defn</u>:

Modular fn. = weakly modular fn., + mero. at ∞.

Mod. form = hol. weakly mod. + hol. at ∞.

cusp form = mod form vanishing at ∞.

f modular form

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}$$

$\Gamma^* := \Gamma - \cancel{\{(0,0)\}}\ \{0\}$ 9

Given $\Gamma \in \mathcal{R}$, define

$$\hat{G}_k(\Gamma) = \sum_{\gamma \in \Gamma^*} \frac{1}{|\gamma|^{2k}} \quad k>1$$

$$\hat{G}_k(\lambda\Gamma) = \lambda^{-2k} G_k(\Gamma)$$

$\cancel{G_k(}$

$\cancel{G_k(z) =}$

Let $\cancel{G_k(\omega_1, \omega_2) = \omega_2^{2k} \tilde{G}_k(\omega_1, \omega_2)}$.

Then $\cancel{G_k(\omega_1, \lambda\omega_1, \lambda\omega_2)}$

$\cancel{= \omega_2^{2k} \lambda^{2k} G_k(}$

Define:

$$G_k(\omega_1, \omega_2) = \tilde{G}_k(\Gamma(\omega_1, \omega_2))\, \omega_2^{2k}.$$

$$G_k(\lambda\omega_1, \lambda\omega_2) = (\omega_2\lambda)^{2k} \tilde{G}_k(\Gamma(\lambda\omega_1, \lambda\omega_2))$$

$$= \omega_2^{2k} \tilde{G}_k(\omega_1, \omega_2)$$

$$= G_k(\omega_1, \omega_2)$$

So for any $z \in H$, makes sense to define 10

$$G_k(z) = G_k(z, 1) = \sum_{(m,n) \neq (0,0)} \frac{1}{(mz+n)^{2k}} \quad \text{for } k>1.$$

$$\boxed{G_k(\omega_1, \omega_2) = G_k(\omega_1/\omega_2)}$$

G_k is the Eisenstein series of weight $2k$.

<u>Thm:</u> $G_k : H \to \mathbb{C}$ is a modular form of weight $2k$.

<u>Pf:</u> ~~Check holomor~~ $\Gamma_z = \langle 1, z \rangle$

$$G_k\left(\frac{az+b}{cz+d}\right) = \sum_{\gamma \in \Gamma_z^*} \frac{1}{|\gamma|^{2k}}$$

$$\sum_{(m,n) \neq (0,0)} \left(m \frac{(az+b)}{(cz+d)} + n \right)^{-2k}$$

$$= (cz+d)^{2k} \sum_{(m,n) \neq (0,0)} \left(m(az+b) + n(cz+d) \right)^{-2k}$$

$$= (cz+d)^{2k} G_k(z).$$

Claim:

$$\sum_{\substack{\cancel{\gamma \in \Gamma^*} \\ (m,n)\neq(0,0)}} \frac{1}{(mz+n)^{2k}} \text{ converges}$$

normally on D

$$|mz+n|^2 = m^2|z|^2 + 2mn\,\mathrm{Re}(z) + n^2$$

$$\geq m^2 - mn + n^2$$

$$= m^2|\rho|^2 + 2mn\,\mathrm{Re}(\rho) + n^2$$

$$= |m\rho + n|^2$$

if $mn \geq 0$

[what if $mn < 0$?] ??

So convergence on D is unif.

hence normal.

$\therefore$ convergence is normal on $g \cdot D$ $\forall\ g \in PSL_2(\mathbb{Z})$.

Since $\{gD \mid g \in PSL_2(\mathbb{Z})\}$ covers H, convergence is normal on H.

To calculate value at ∞,

$$\lim_{r\to\infty} G_k(ir)$$

$$= \lim_{r\to\infty} \sum_{(m,n)\neq(0,0)} \frac{1}{(m\,ir+n)^{2k}}$$

$$= \sum_{(m,n)\neq(0,0)} \lim_{r\to\infty} \frac{1}{(m\,ir+n)^{2k}}$$

$$= \sum_{n\neq 0} \frac{1}{n^{2k}} = 2\zeta(2k).$$

$$g_2 = 60G_2 \qquad g_3 = 140G_3$$

$$g_2(\infty) = \frac{3}{4}\pi^4$$

$$g_3(\infty) = \frac{8}{27}\pi^6$$

$\Delta = g_2^3 - 27g_3^2$ is ~~modu~~ a cusp form of wt 12