1

Functional eqns. of the type:

$$y = z\varphi(y(z))$$

with non-neg. Taylor coeff.

<u>Thm:</u> $\varphi$ is hol. in a nbd of 0, $\varphi(0) \neq 0$, $R \leq +\infty$ the radius of conv. of $\varphi$ at 0, and

$$\lim_{x \to R^-} \frac{x\varphi'(x)}{\varphi(x)} > 1.$$

Then $\exists!\ \tau \in (0, R)$ such that

$$\boxed{\frac{\tau\varphi'(\tau)}{\varphi(\tau)} = 1}$$ - char. eqn.

The formal solution of $y = z\varphi(y(z))$ is analytic around 0, and its coeffs. have exp. growth $a_n \bowtie \left(\frac{1}{\rho}\right)^n$. where

$$\rho = \frac{\tau}{\varphi(\tau)} = \frac{1}{\varphi'(\tau)}.$$

2

Formally,

$$y(z) = \sum_{n=0}^{\infty} a_n z^n$$

has uniquely determined coefficients, and $a_0 = 0$.

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$$z = \frac{y}{\varphi(y)} =: \psi(y).$$

<u>Inverse function thm:</u>

If $\psi$ is analytic around $y_0$, $\psi(y_0) = z_0$, $\psi'(y_0) \neq 0$, then for some nbd. $\Omega_0$ of $z_0$, $\exists$ hol. $y(z)$ such that $\psi(y(z)) = z$ $\forall\ z \in \Omega_0$, $y(z_0) = y_0$.

3

$\psi(y_0) = z_0$

Pf: Let $\sigma(z) = \frac{1}{2\pi i} \int_{\partial B_\varepsilon(y_0)} \frac{\psi'(y)}{\psi(y)-z}\, dy$

Residue thm. $\Rightarrow$

$\sigma(z_0) = 1$

$\sigma(z)$ is locally const. in $z$, and so $\sigma(z) = 1$ for all $z$ in some nhd of $z_0$.

So every value in a nhd $\Omega_0$ of $z_0$ is taken exactly in a nhd. of $y_0$.

$\forall\ z \in \Omega_0$, let $y(z)$ be the unique element of $B_\varepsilon(y_0) \ni$

$\psi(y(z)) = z$.

4

$$\sigma_1(z) = \frac{1}{2\pi i} \int_{\partial B_\varepsilon(y_0)} \frac{\psi'(y)\, y}{\psi(y)-z}\, dy$$

$= y(z)$ analyticity

By analytic inversion, $y(z)$ is an analytic fn. in a nhd. of $0$.

$$\psi(y) = \frac{y}{\varphi(y)}$$

$$\psi'(y) = \frac{\varphi(y) - y\varphi'(y)}{\varphi(y)^2}.$$

put $y=0$, $\psi'(0) \neq 0$.

⑤

$$\cancel{\alpha(x) = \frac{x\varphi'(x)}{\varphi(x)} = 0 \text{ at } x=0}$$

$$\cancel{> 1 \text{ as } x \to R^-}$$

$$\cancel{\alpha'(x) = \frac{(x\varphi''(x) + \varphi'(x))\varphi(x)}{\varphi(x)^2} - \frac{x\varphi'(x)^2}{\varphi(x)^2}.}$$

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$$\frac{x\varphi'(x)}{\varphi(x)} = 1$$

$$\varphi(x) = x\varphi'(x)$$

$$\varphi_0 + \varphi_1 x + \varphi_2 x^2 + \cdots$$

$$= x\varphi_1 + 2x^2\varphi_2 + 3x^3\varphi_3 + \cdots$$

⑥

$\iff$

$$\varphi_0 = x^2\varphi_2 + 2x^3\varphi_3 + 3x^4\varphi_4$$

RHS is increasing, so soln. is unique.

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<u>Singular inversion</u>

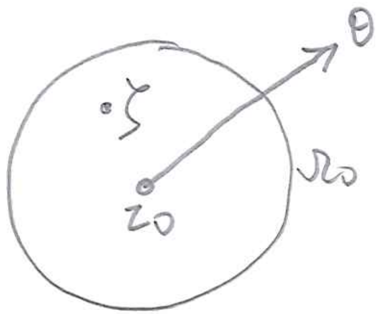
$\psi$ analytic around $y_0$, and $\psi'(y_0) = 0$, $\psi''(y_0) \neq 0$.

$$\sigma(z) = \frac{1}{2\pi i}\int_{B_\varepsilon(y_0)} \frac{\psi'(y)\,dy}{\psi(y) - z}$$

$$\sigma(z_0) = 2.$$

$\exists\ \Omega_0$ open nhd. of $z_0$ such that $\forall\ z \in \Omega_0$, $\exists$ exactly 2 points, $y_1$ & $y_2$ $\ni$ $\psi(y_i) = z$.

Take $\Omega_0^\theta$ = slit nbd. of $z_0$. (7)




for any $\zeta \in \Omega_0^\theta$, take $\eta_1, \eta_2$ such that $\psi(\eta_i) = \zeta$.

Since $\psi'(\eta_i) \neq 0$, analytic inversion applies, and can construct $y_1$ & $y_2$ in a nbd. of $\zeta$ such that $\psi(y_1(z)) = z$ and $\psi(y_2(z)) = z$.

→ gives rise to well-defined hol. fns $y_1, y_2 : \Omega_0^\theta \to \mathbb{C}$

$\ni \psi(y_i(z)) = z \quad \forall \ z \in \Omega_0^\theta$.

(8)

The functions $y_i$ cannot be extended to a nbd. of $z_0$.

$$\cancel{z = \psi(y) = y_0 + (y-y_0)^2 \psi''(y_0)}$$
$$\cancel{+ o((y-y_0)^2)}$$

$$\cancel{z = \psi(y_i(z))} \ 1 = (\psi \circ y_i)'(z)$$
$$= \psi'(y_i(z))\, y_i'(z) = 0 .$$

LHS = 1 at $z = z_0$; RHS = 0

$\Rightarrow \Leftarrow$.

$$\psi'(y_0) = \psi''(y_0) = \cdots = \psi^{(k+1)}(y_0) = 0$$

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If $\psi(y) = z_0 + (y-y_0)^k \alpha(y)$

then, $\exists \ y_1, \dots, y_k : \Omega_0^\theta$

$\ni \psi(y_i(z)) = z$, all will have sing at $z_0$.