(Ordered) binary trees

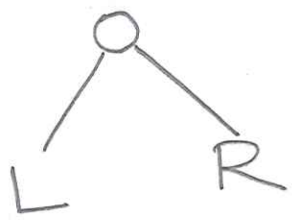
→ root node

→ each node is either a leaf or has a left child and a right child.

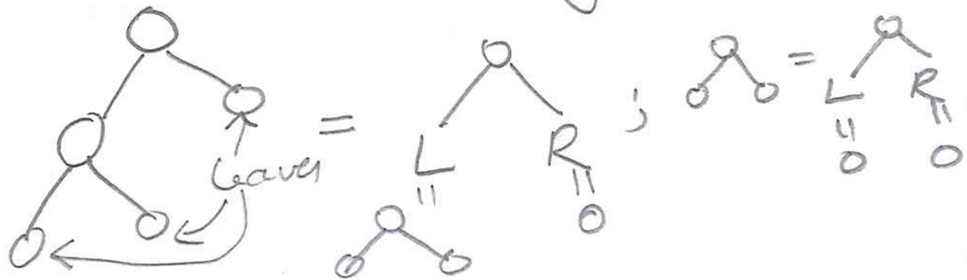
Recursive construction:

A binary tree is one of

o, or



where L & R are binary trees.



leaf $\Longleftrightarrow$ no children

internal node $\Longleftrightarrow$ not a leaf.

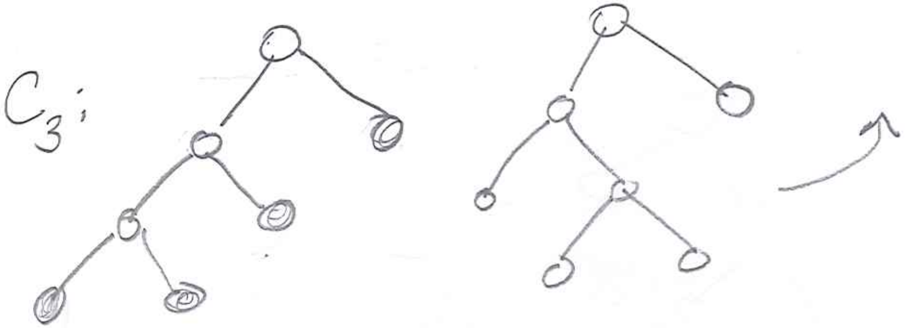
__Problem:__ Count the no. of C_n. binary trees with n internal nodes.

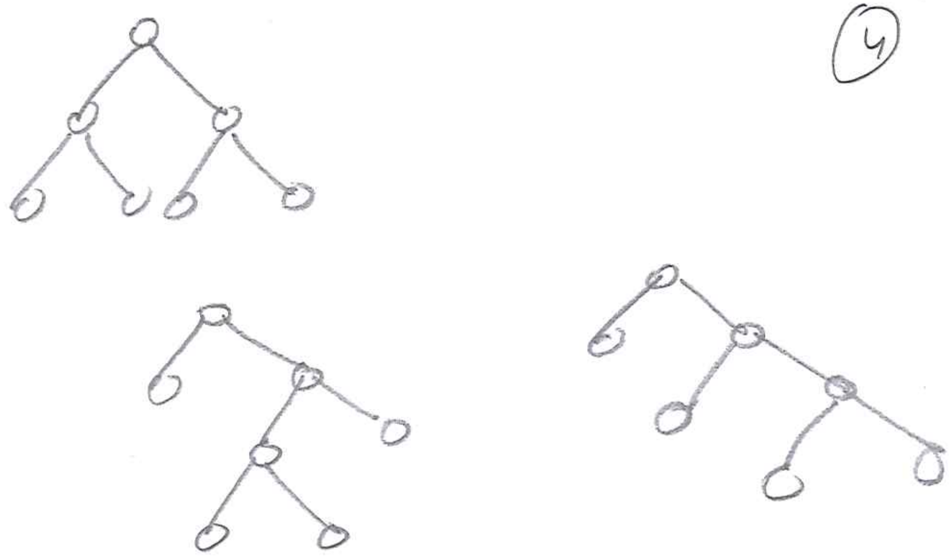
C_0 :

C_1 :

C_2 :

C_3 :





$$C_n = \sum_{i+j=n-1} C_i C_j .$$

__Defn:__ $wt(T)$ = no. of internal nodes

$$y(z) = \sum_{T} z^{wt(T)}$$

~~Given T, define a bije~~

$\mathcal{T}$ = set of all binary trees.

Recursive specifications

$$\rightsquigarrow \underbrace{y = z\varphi(y)}_{\text{functional eqn.}}$$

e.g. binary trees
$$\varphi(y) = (1+y)^2$$

Motzkin tree
$$\varphi(y) = 1+y+y^2$$

Suppose $y = \sum_{n=0}^{\infty} a_n z^n$.

$$\varphi = \sum_{n=0}^{\infty} \varphi_n z^n.$$

$$\sum_{n=0}^{\infty} a_n z^n = z \sum_{n=0}^{\infty} \varphi_n \left(\sum_{k=0}^{\infty} a_k z^k\right)^n$$

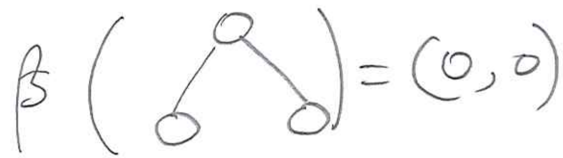
$$\beta : \mathcal{T} \xleftrightarrow{\text{bij}} \{\circ\} \sqcup \mathcal{T}^2$$

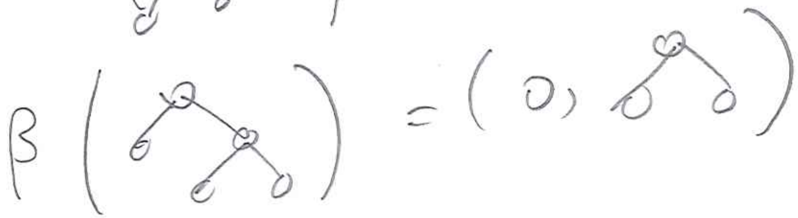
$$\circ \longrightarrow \circ$$

$\longrightarrow (L, R)$

Recursive spec $\Longleftrightarrow$ the above is a bijection

Example:





$$\sum_{T\in\mathcal{T}} \cancel{wt}\, z^{wt(T)}$$

$$= 1 + \sum_{(T_1,T_2)\in\mathcal{T}} z^{wt(T_1)+wt(T_2)+1}$$

$$= 1 + z \sum_{(T_1,T_2)\in\mathcal{T}} z^{wt(T_1)} z^{wt(T_2)}$$

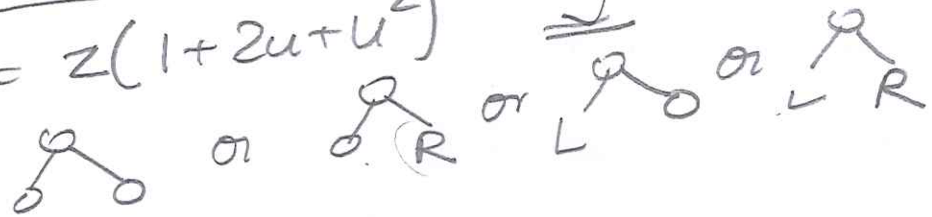
$$= 1 + z\Big(\sum_{T_1\in\mathcal{T}} z^{wt(T_1)}\Big)\Big(\sum_{T_2\in\mathcal{T}} z^{wt(T_2)}\Big)$$

$y = 1 + zy^2$. $\mathcal{T}^* = \mathcal{T} - \{o\}$

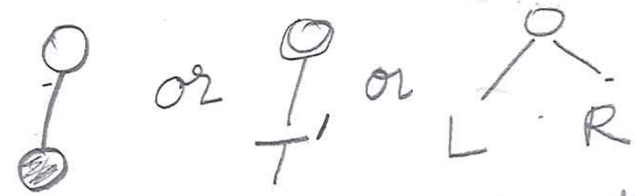
Let $u = y - 1$

$u = z(u+1)^2$

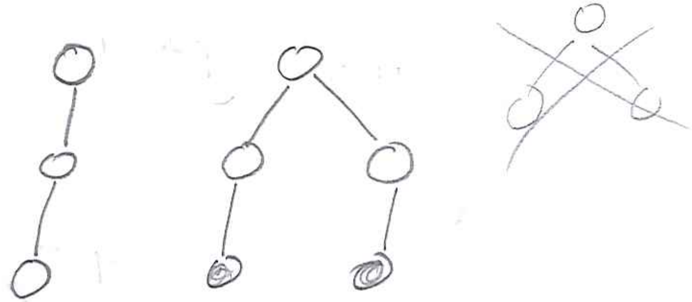
$u = z(1+2u+u^2)$ $\underline{\mathcal{T}^*}$

$T =$ 

Motzkin trees: $\mathcal{M}^*$

$T =$ 

$wt = 1$ $wt(T') + 1$ $wt(L) + wt(R) + 1$



$$y = z + zy + zy^2$$

$$y = z(1 + y + y^2).$$

$a_0 = 0$

$a_1 = \varphi_0$

$a_2 = \varphi_1 a_1$

$a_3 = \varphi_1 a_2 + \varphi_2 a_1$

$a_4 = \varphi_1 a_3 + \varphi_2$

FORMAL SOLN

$$\boxed{a_n = \sum_{j_1 + \cdots + j_i + 1 = n} \varphi_i a_{j_1} \cdots a_{j_i}}$$

→ a_n's are uniquely determined by φ_i's.

→ $a_n \geq 0$ if $\varphi_n \geq 0$

Thm: Let φ be hol. in a nhd. of 0, with non-neg. Taylor coeffs., $\phi'(0) \neq 0$. Let R $(\leq \infty)$ be the radius of convergence of the power series of φ around 0. Suppose

$$\lim_{x \to R^-} \frac{x\varphi'(x)}{\varphi(x)} > 1.$$

Then $\exists!$ $\tau \in (0, R)$ such that $\frac{\tau\varphi'(\tau)}{\varphi(\tau)} = 1$

The formal solution ~~to~~ ~~analytic at 0, and~~

$y = z\varphi(y)$

is analytic at 0, $y(z) = \sum_{n=0}^{\infty} a_n z^n$

$a_n \bowtie \left(\frac{1}{\rho}\right)^n$, $\rho = \frac{\tau}{\varphi(\tau)} = \frac{1}{\varphi'(\tau)}$