①

$$f(z)=\sum_{n=0}^{\infty} a_n z^n \quad \text{meromorphic in } \mathbb{C}$$

$$\frac{1}{2\pi i}\int_{\partial B_R} \frac{f(z)}{z^{n+1}}\,dz$$

$$= a_n + \sum_{\substack{c\in \text{poles}(f)\\ \text{in } B_R}} \operatorname{res}_c\left(\frac{f(z)}{z^{n+1}}\right)$$

Take limit as $R\to\infty$.

If LHS $\to 0$ as $R\to\infty$, get an infinite series exp. for a_n

$$a_n = -\sum_{c\in \text{poles}(f)} \operatorname{res}_c\left(\frac{f(z)}{z^{n+1}}\right).$$

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Bernoulli numbers:

$$= \frac{z}{z+\frac{z^2}{2!}+\frac{z^3}{3!}}$$

$$B(z) = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n = \frac{z}{e^z-1}$$

②

Removable singularity at 0

$$B(0) = 1 = B_0$$

$$B(1) = \lim_{z\to 0}\frac{d}{dz}B(z)$$

$$= \lim_{z\to 0}\frac{(e^z-1)-ze^z}{(e^z-1)^2}$$

$$= \lim_{z\to 0}\frac{e^z-ze^z-1}{(e^z-1)^2}$$

$$= \lim_{z\to 0}\frac{\cancel{e^z}-\cancel{e^z}-ze^z}{2e^z(e^z-1)}$$

$$= -\frac{1}{2} = B_1.$$

$$B(z)+\frac{z}{2} = \frac{z}{e^z-1}+\frac{z}{2}$$

$$= \frac{2z+z(e^z-1)}{2(e^z-1)} = \frac{z(e^z+1)}{2(e^z-1)}$$

$$= \frac{z}{2}\,\frac{e^{z/2}+e^{-z/2}}{e^{z/2}-e^{-z/2}} = \frac{z}{2}\coth\frac{z}{2}.$$

$$B(z) = \frac{z}{2}\left(\coth\frac{z}{2} - 1\right)$$

$B_{2m+1} = 0$ for $m > 0$

$\operatorname{sgn}(B_{2m}) = (-1)^{m+1} \quad m > 0$

Discovered by Jakob Bernoulli

$$\sum_{k=1}^{n} k^c = \sum_{k=0}^{c} \frac{|B_k|}{k!}\, c^{(k-1)}\, n^{c-k+1}.$$

$$c^{(r)} = c(c-1)\cdots(c-r+1)$$

$$c^{(0)} = 1$$

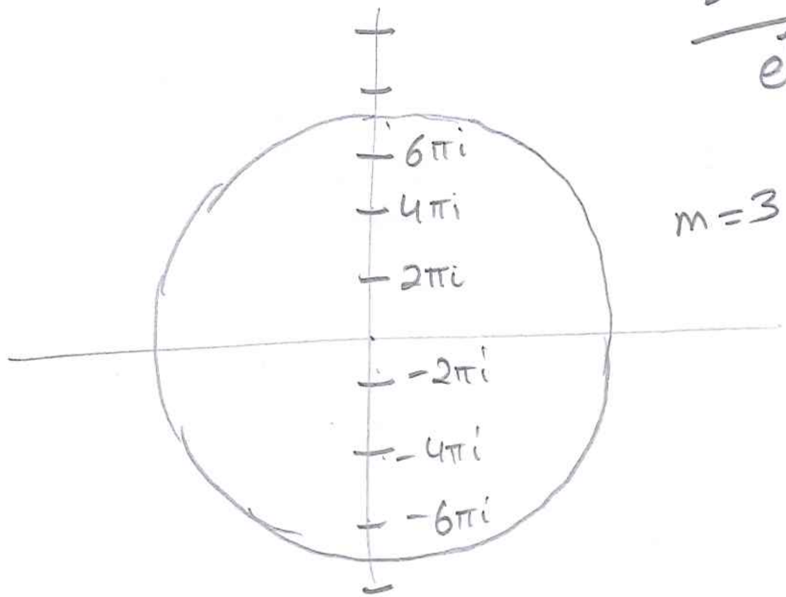
$$c^{(-1)} = \frac{1}{c+1}$$

$$B(z) = \frac{z}{e^z-1}.$$

$$\frac{1}{2\pi i}\int_{\partial B_{(2m+1)\pi i}} \frac{B(z)}{z^{n+1}}\,dz$$

$$\frac{z}{}$$

$$\frac{z(z-2\pi i k)}{e^z-1}$$



$m=3$

$$= \frac{B_n}{n!} + \sum_{\substack{k=-m\\k\neq 0}}^{+m} \operatorname{res}_{2\pi i k}\frac{z}{(e^z-1)\,z^{n+1}}$$

$$= \frac{B_n}{n!} + \sum_{\substack{k=-m\\k\neq 0}}^{+m} (2\pi i k)^{-n}.$$

(5)

Rational gen. fns. $\Leftrightarrow$ finite recurrences

Eg: $f_n = f_{n-1} + f_{n-2}$

Finite recurrence: with m terms

$$a_n = \alpha_1 a_{n-1} + \alpha_2 a_{n-2} + \dots + \alpha_m a_{n-m}$$

$$\alpha_1, \dots, \alpha_m \in \mathbb{C}.$$

$$\sum a_n z^n = Q(z) + \frac{N(z)}{D(z)} \leftarrow \text{monic deg } r.$$

$$D(z) \sum a_n z^n = Q(z) D(z) + N(z)$$

$$\begin{array}{c} \| \\ d_0 + d_1 z + \dots + d_r z^r \end{array}$$

$$\sum_{k=0}^{r} d_k a_{n-k} = Q(z) D(z) + N(z)$$

$$= 0 \text{ for } n >> 0 .$$

(6)

Thm: $\sum_{n=0}^{\infty} a_n z^n$ is a rational function iff $\exists\, m > 0$, $N > 0$ and $\alpha_1, \dots, \alpha_m \in \mathbb{C}$ such that

$$a_n = \alpha_1 a_{n-1} + \dots + \alpha_r a_{n-m}$$

for $n > N$.

Pf: $a_n = 0$ if $n < 0$

$$\vec{a}_n^m = \begin{pmatrix} a_n \\ a_{n-1} \\ \vdots \\ a_{n-m+1} \end{pmatrix}$$

$\{\vec{a}_n^m\}_{n=0}^{\infty}$ is a sequence of vectors

$$\vec{a}_0^m = \begin{pmatrix} a_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = a_0 \vec{e}_1$$

$$\vec{a}_{n+1}^m = \begin{pmatrix} a_n \\ a_{n-1} \\ \vdots \\ a_{n-m+2} \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_m \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ 0 & \dots & 1 & 0 \end{pmatrix}}_{A} \begin{pmatrix} a_{n-1} \\ a_{n-2} \\ \vdots \\ a_{n-m} \end{pmatrix}$$

① Taking $\lim_{m\to\infty}$, the contour integral goes to 0.

<u>Get:</u>

$$B_n = (-n!) \sum_{k\in\mathbb{Z}-\{0\}} (2\pi i k)^{-n}$$

Suppose $n = 2m+1$

$$B_{2m+1} = (-(2m+1)!)$$

$$\sum_{k=\phi}^{\infty} \qquad = 0$$

Suppose $n = 2m$.

$$B_{2m} = (-1)(2m)! \sum_{k=1}^{\infty} (-1)^m 2 (2\pi k)^{-2m}$$

$$= (-1)^{m-1} 2^{1-2m} \pi^{-2m} \sum_{k=1}^{\infty} \frac{1}{k^{2m}} = \zeta(2m)$$

$$\zeta(2m) = (-1)^{m-1} 2^{2m-1} \pi^{2m} \frac{B_{2m}}{(2m)!} \quad ⑧$$

<u>Rationality of B_{2n}</u>

$$\sum \frac{B_n}{n!} z^n = \frac{z}{z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots}$$

$$= \frac{1}{1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots}$$

$$\left(1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots\right) \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n = 1$$

Equating coeffs. on both sides:

$$\sum_{k=0}^{n} \frac{1}{(k+1)!} \frac{B_{n-k}}{(n-k)!} = \delta_{0,n}$$

(9) we took

$$A = \begin{pmatrix} \alpha_1 & \cdots & \cdots & \alpha_m \\ 1 & 0 & \cdots & 0 \\ & & & \\ 0 & \cdots & 1 & 0 \end{pmatrix}$$

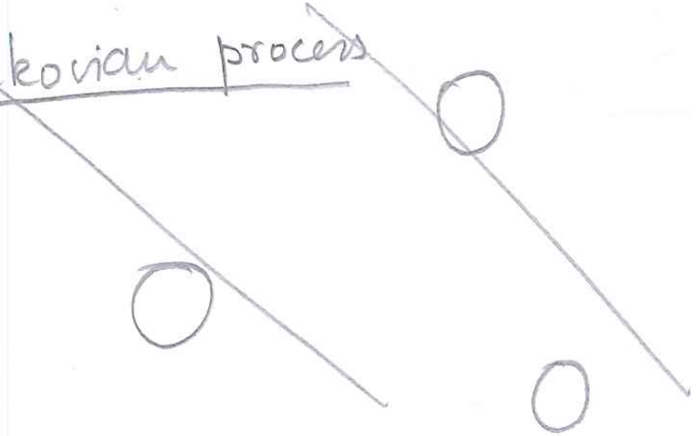
can take any <u>general A,</u>

and the same reasoning holds.

If $\vec{a}_n^m = A\vec{a}_{n-1}^m$ for $n \geq 0$

then $\sum a_n z^n$ is rational.

<u>Markovian process</u>



$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1-z & -z \\ -z & 1 \end{pmatrix}$$

$-z^2 + 1 - z$

$$\vec{a}^{m}_{n+1} = A\vec{a}^{m}_{n-1} = A^2\vec{a}^{m}_{n-2}$$

$$= \dots = A^n \vec{a}^{m}_0$$

$$= a_0 A^n \vec{e}_1$$

$$\sum_{n=0}^{\infty} \vec{a}^{m}_n z^n = a_0 \sum_{n=0}^{\infty} (A^n \vec{e}_1) z^n$$

<u>Claim:</u> $\sum_{n=0}^{\infty} A^n z^n = (I - Az)^{-1}$

as matrix formal power series.

<u>Pf:</u> Multiply both sides by $(I-Az)$.

$$\sum_{n=0}^{\infty} \vec{a}^{m}_n z^n = a_0 (I-Az)^{-1} \vec{e}_1$$

So $\sum_{n=0}^{\infty} a_n z^n = a_0 \vec{e}_1^{\,T} \cdot (I-Az)^{-1} \vec{e}_1$

$A_{m \times m}$.

$$A^{-1} = \frac{A^{\circledast}}{\det(A)}$$

the entries of $A^{\circledast}$ are homogeneous polynomials of deg $(m-1)$ in the entries of A.

$$\text{So } \sum_{n=0}^{\infty} a_n z^n = \frac{a_0 \vec{e}_1^{\,T} (I-Az)^{\circledast} \vec{e}_1}{\det(I-Az)}$$

$$= \frac{p(z)}{z^m \chi_A(z^{-1})} \qquad z^m \chi_A(z^{-1})$$

(char. poly).

$p(z)$ = poly of degree $\leq m-1$