①

$$\sum_{n=0}^{\infty} a_n z^n = f(z) = \frac{N(z)}{D(z)}$$

$$= Q(z) + \sum_{(\alpha,r)} \frac{c_{\alpha,r}}{(z-\alpha)^r}$$

α runs over poles

$1 \le r \le$ order of pole α.

$$\frac{1}{(z-\alpha)^r} = \sum_{n=0}^{\infty} (-1)^n \frac{r(r+1)\dots(r+n-1)}{n!}\alpha^{-r-n} z^n$$

$\frac{r(r+1)\dots(r+n-1)}{n!} = \binom{r+n-1}{r-1}$

poly in n of degree $r-1$

① $|\alpha|$ - minimum, then

② k - max. → pole of highest order closest to 0.

$$\frac{c_{k,\alpha}\,\alpha^{-(n+k_i)}\, n^{(k-1)}}{(k-1)!}$$

②

$p(n,k)$ = no. of partitions of n with parts $\le k$

$$= \left\{(\lambda_1,\dots,\lambda_\ell) \;\middle|\; \begin{matrix} k \ge \lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_\ell > 0 \\ \lambda_1 + \dots + \lambda_\ell = n \end{matrix}\right\}$$

$$\sum_{n=0}^{\infty} p(n,k) z^n = \prod_{m=1}^{k} \frac{1}{1-z^m}$$

$$= \prod_{m=1}^{k} (1 + z^m + z^{2m} + \dots)$$

$$= \sum_{0 \le i_1, i_2, \dots i_k < \infty} z^{i_1 + 2i_2 + \dots + k i_k}$$

Coeff. of z^n

$$= \#\{(i_1,\dots,i_k) \mid i_1 + 2i_2 + \dots + k i_k = n\}$$

$$(i_1,\dots,i_k) \longleftrightarrow \underbrace{k,k,\dots,k}_{i_k}, \underbrace{k-1,\dots,k-1}_{i_{k-1}}, \dots \underbrace{1,\dots,1}_{i_1}$$

$k=3$ $\quad (2,1) \leftrightarrow (1,1,0)$

$(3,3,1) \leftrightarrow (1,0,2)$ etc.

$$\sum_{n=0}^{\infty} p(n,k) z^n = \prod_{m=1}^{k} \frac{1}{1-z^m}$$

poles $= \{\zeta \in \mathbb{C} \mid \zeta^m = 1 \text{ for some } 1 \le m \le k\}$

<u>multiplicities of poles:</u>

$$(z^m - 1) = \prod_{l=0}^{m-1} \left(z - e^{2\pi i l/m}\right)$$

mult. of $1 = k$

mult. of $-1 = \lfloor k/2 \rfloor$

mult of $e^{2\pi i/3} = \lfloor k/3 \rfloor$

<u>Contribution of the pole $z = 1$:</u>

<u>give asympotic value:</u>

$$\lim_{z \to 1} \left| \frac{(z-1)^k}{\prod_{m=1}^{k}(1-z^m)} \frac{n^{k-1}}{(k-1)!} \right| = \frac{n^{k-1}}{k!(k-1)!}$$

$f : B_R(0) \to \hat{\mathbb{C}}$ meromorphic function with finitely many poles $\alpha_1, \dots, \alpha_m$.

then

$$f(z) = Q(z) + \sum_{(\alpha, r)} \frac{c_{\alpha,r}}{(z-\alpha)^r}$$

$\alpha \in \{\alpha_1, \dots, \alpha_m\}$.

$1 \le r <$ order of pole at α

Coefficients of $Q(z) = \sum b_n z^n$.

$$b_n = \frac{1}{2\pi i} \int_{\partial B_r(0)} \frac{Q(z)\,dz}{z^{n+1}}$$

$0 < r < R$,

$$\le \sup_{|z|=R} |Q(z)|\, r^{-n}$$

$$a_n = \sum_{j=1}^{m} \pi_j(n) \alpha_j^{-n} + O(r^{-n})$$

$\forall\, r < R$.

where π_j is a poly in n of degree $k_j - 1$, k_j - order of α_j

$M(f;r)/r^n$ $\qquad |r| > |\alpha_i|$

$$\Rightarrow \frac{1}{2\pi i}\int_{\partial B_r(0)} \frac{f(z)}{z^{n+1}}\,dz$$

$$= a_n + \sum_{\alpha \text{ pole}} \mathrm{Res}_{\alpha} \frac{f(z)}{z^{n+1}}.$$

what is this in terms of n, α and $c_{\alpha,r}$? ??

$$\frac{1}{1-z^3} = 1+z^3+z^6+z^9+\cdots$$

$$a_n = \begin{cases} 0 & \text{if } n \equiv 1,2 \bmod 3 \\ 1 & \text{if } n \equiv 0 \bmod 3 \end{cases}$$

Qn: What is the "period" of

$$\frac{1}{1-z^2}\cdot\frac{1}{1-z^3} = (1+z^2+z^4+\cdots)(1+z^3+\cdots)$$

HW $a_n = ??$ depends on $n \pmod 6$.

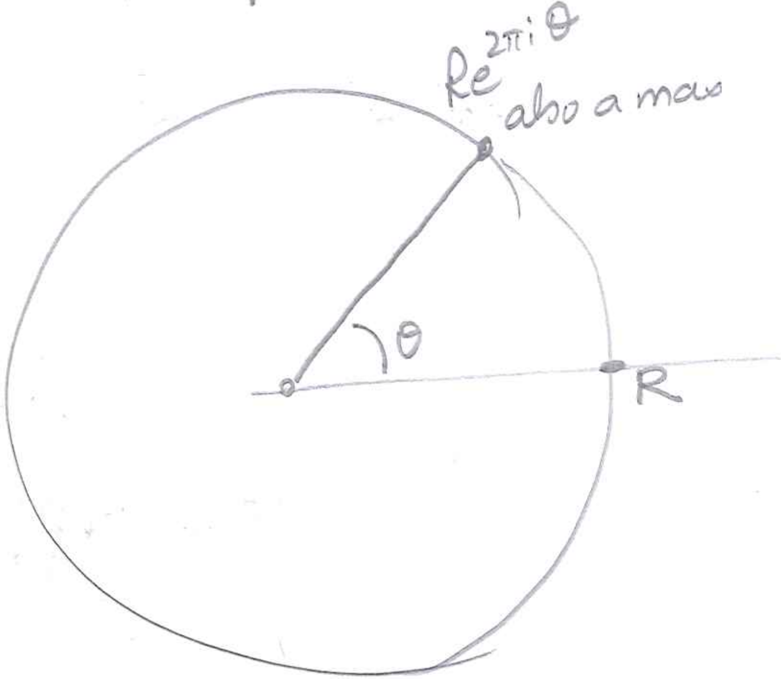
$f(z) = \sum_{n\geq 0} a_n z^n$, $\quad a_n \geq 0$.

$\mathrm{supp}(f) = \{n \geq 0 \mid a_n > 0\}$.

Daffodil lemma: Suppose $f(z) = \sum a_n z^n$ $\boxed{a_n \geq 0}$

Suppose $\exists\ 0<\theta<1$ such that, for some $R>0$,

$$|f(Re^{2\pi i\theta})| = |f(R)|$$



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$$\left|\sum_{n=0}^{\infty} a_n (Re^{2\pi i\theta})^n\right| = \sum a_n R^n$$

$$\left|\sum_{n\in supp(f)} a_n (Re^{2\pi i\theta})^n\right| = \sum a_n R^n.$$

$\forall\, n \in supp(f)$

$$e^{2\pi i n\theta} = e^{2\pi i \alpha}$$

If $n_1 = \min supp(f)$.

$\alpha = n_1\theta + k$.

If $supp(f) \neq \{n_1\}$,

$\alpha = n_2\theta + k'$

$(n_2 - n_1)\theta \in \mathbb{Z}$

$\Rightarrow \theta \in \mathbb{Q} = \frac{p}{q}, (p,q)=1$

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Claim:

$supp(f) \subseteq n_1 + \mathbb{Z}_{\geq 0}\, q$

Pf: Suppose $n \in supp(f)$.

Then $e^{2\pi i n \frac{p}{q}} = e^{2\pi i n_1 \frac{p}{q}}$

$\Leftrightarrow (n - n_1)\frac{p}{q} \in \mathbb{Z}$.

$\Leftrightarrow (n-n_1)$ is a multiple of q.

QED.