1

- $a_n \bowtie K^n$

"a_n has exponential order K^n"

$\iff \limsup_{n\to\infty} |a_n|^{1/n} = K$

$\iff \sum_{n=0}^{\infty} a_n z^n$ has radius of convergence $\frac{1}{K}$.

- If $\sum_{n=0}^{\infty} a_n z^n$ is entire,

$a_n R^n \to 0 \quad \forall R > 0$

Example: $\frac{R^n}{n!} \to 0 \quad \forall R>0$

$\Uparrow$

$\exp(z)$ is entire.

If $a_n \bowtie K^n$, write

$a_n = K^n \theta(n)$ ← subexponential factor

$\limsup_{n\to\infty} |\theta(n)|^{1/n} = 1$

2

Pringsheim's theorem If $a_n \geq 0$, and $\sum_{n=0}^{\infty} a_n z^n$ has radius of conv. R, then R is a singularity.

Example: $\sum_{n=0}^{\infty} f_n z^n = \frac{1}{1-z-z^2}$

fibonacci

$f_0 = 1 \quad f_1 = 1, \quad f_n = f_{n-1} + f_{n-2}$
$n \geq 2$.

poles at $z = \frac{-1 \pm \sqrt{5}}{2}$

The pole closest to 0 is $\frac{\sqrt{5}-1}{2}$

$f_n \bowtie \left(\frac{-1+\sqrt{5}}{2}\right)^{-n} = \left(\frac{1+\sqrt{5}}{2}\right)^n$

golden ratio.

$$C_n = \sum_{k+l=n-1} C_k C_l .$$

$$f(z) = \sum_{n=0}^{\infty} C_n z^n$$

$$= \frac{1-\sqrt{1-4z}}{2z}$$

$$C_n \bowtie 4^n$$

Saddle point bounds

$$|a_n| = \left| \frac{1}{2\pi i} \int_{\partial B_r(0)} \frac{f(z)}{z^{n+1}}\, dz \right|$$

$$\le \frac{\max_{|z|=r} |f(z)|}{r^n} =: \frac{M(f;r)}{r^n}.$$

by Cauchy integral formula.

If $a_n \geq 0$,

$$\sup_{|z|=r} \left| \sum_{n=1}^{m} a_n z^n \right| \le \sum_{n=1}^{m} a_n r^n$$

So, $M(f;r) = f(r)$.

Thm: $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic in $|z| < R$, and

$$M(f;r) := \sup_{|z|=r} |f(z)|$$

then $|a_n| \le \frac{M(f;r)}{r^n} \quad \forall\ 0<r<R$

If $a_n \geq 0$, then

$$|a_n| \le \frac{f(r)}{r^n} \quad \forall\ 0<r<R$$

Cor: $|a_n| \le \inf_{0<r<R} \frac{M(f;r)}{r^n}$.

for $a_n \geq 0$ $\quad a_n \le \inf_{0<r<R} \frac{f(r)}{r^n}$.

To find a min.,

set $\frac{d}{dr} \frac{f(r)}{r^n} = 0$.

i.e. $f'(r) r^n - n f(r) r^{n-1} = 0$

$$(*) \quad \text{or} \quad n = r \frac{f'(r)}{f(r)} = r \frac{d}{dr} \log f$$

Example: $f(z) = \exp(z) = \sum \frac{z^n}{n!}$

$$\frac{1}{n!} \leq \frac{\exp(r)}{r^n} \quad \forall\, r > 0$$

(*) gives $n = r$

$$\frac{1}{n!} \leq \left(\frac{e}{n}\right)^n$$

Stirling $n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$

What happens for Bell nos?

$f(z) = e^{e^z - 1}$

$$\frac{B_n}{n!} \leq \frac{e^{(e^r - 1)}}{r^n}$$

(*) gives: $n = r \frac{d}{dr}(e^r - 1)$

$= r e^r$

$\log n = r + \log r$

$r = \log n - \log r$

$= \log n - \log(\log n - \log r)$

$\overset{??}{=} \log n - \log \log(n) + o(\log\log n)$

Rational generating functions:

$$f(z) = \sum a_n z^n = \frac{N(z)}{D(z)}$$

$N(z), D(z) \in \mathbb{C}[z]$

no common roots.

poles = roots of $D(z)$.

$$\sum a_n z^n = Q(z) + \sum_{(\alpha,r)} \frac{c_{\alpha,r}}{(z-\alpha)^r} \quad (\dagger)$$

α runs over the poles of f,

if α has degree k, $0 < r \leq k$.

Coeff of z^n in $\frac{1}{(z-\alpha)^r}$

is $\frac{1}{n!} \frac{d^n}{dz^n} \frac{1}{(z-\alpha)^r}\Big|_{z=0}$

$$= \frac{1}{n!}(-1)^n \frac{r(r+1)\dots(r+n-1)}{(z-\alpha)^{r+n}}\Big|_{z=0}$$

$$= \frac{(-1)^r}{\alpha^{r+n}} \binom{n+r-1}{n}$$

$$\binom{n+r-1}{n} = \frac{(n+r-1)(n+r-2)\dots(n+1)}{(r-1)!}$$

polynomial in n of degree $(r-1)$ and leading term $\frac{1}{(r-1)!}$

Coeff. of z^n in $\frac{1}{(z-\alpha)^r}$ is

a polynomial in n of deg. $r-1$

and leading coeff $\frac{1}{(r-1)!}$

Applying to $(\dagger)$:

$$a_n = \sum_\alpha \Pi_\alpha(n)\alpha^{-n} \quad (*)$$

where $\Pi_\alpha(n)$ is a polynomial in n, of degree $\mathrm{ord}_f(\alpha) - 1$

order of pole α.

for $n > \deg N - \deg D$.

Asymptotics:

$$a_n \sim b_n$$

means: $\exists\ 0 < k < K < \infty \ \ni$

$$k < \frac{a_n}{b_n} < K$$

for suff. large n.

To get the asymptotic from (*), Take only terms α with min. absolute value, and only leading terms of Π_α;

If $\alpha_1 \ldots \alpha_m$ are the min. abs. value poles of f, and $k_1, \ldots, k_m$ are their orders gives

$$a_n \sim \sum_{i=1}^{m} \frac{\lim_{z\to\alpha}(z-\alpha)^{k} f(z)}{(k-1)!} \alpha^{-(n+k_i)} n^{(k_i-1)}$$

Can only take those with max. k_i.

Example:

$$f(z) = \frac{1}{(1-z^3)^2(1-z^2)^3(1-\frac{z^2}{2})}$$

1 has mult. 5
ω, ω^2 have mult. 2] poles of smallest abs. value
-1 has mult. 3

$$a_n \sim \frac{1}{4!}\ \frac{1}{9}\cdot\frac{2}{8}\, n^4$$

$$\lim_{z\to 1} \frac{z^3-1}{z-1} = 3$$

$$\frac{3z^2}{1}$$

$$1-\frac{1}{2}$$

$$= \frac{n^4}{24\times4\times9}$$

$$= \frac{n^4}{864}.$$

$$\sum_{n=0}^{\infty} P_{n,k} z^n = \prod_{m=1}^{k} \frac{1}{1-z^m}$$

Can do similar analysis.