Generating functions

1

$\{a_n\}$ sequence

ordinary

OGF: $\sum_{n=0}^{\infty} a_n z^n$ "unlabeled"

expo.

EGF: $\sum_{n=0}^{\infty} \frac{a_n z^n}{n!}$ "labeled objects"

Example 1 Fibonacci

$a_0 = 1, \quad a_1 = 1$

$a_n = a_{n-1} + a_{n-2}.$

$$F(z) = \sum_{n=0}^{\infty} a_n z^n = \frac{1}{1 - z - z^2}$$

$$= 1 + (z+z^2) + (z+z^2)^2 + \cdots$$

$$(z+z^2)^n = (z+z^2)(z+z^2)\ldots(z+z^2)$$

Example 2: Catalan

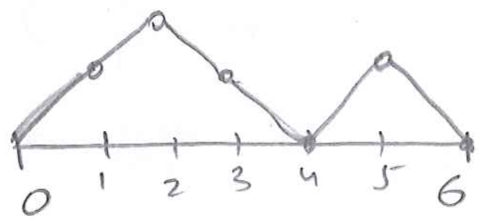
2

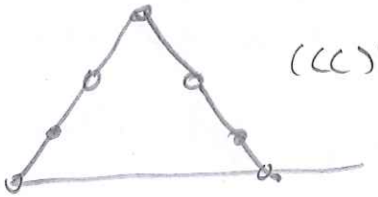
Well-formed parentheses of length $2n$.

n	
0	ϕ.
1	()
2	(()), ()()
3	((())), (())(), ()()(), ()(()),

Dyck paths:

(())()





((()))

(→

) →

Well formed parentheses: of length 2n

(3)

an expression of the form:

$$a_1 a_2 \cdots a_{2n}$$

where $a_i \in \{ (,) \}$

Such that

→ no. of "(" = no. of closed ")"

$= n$.

→ no. of "(" in $a_1 \ldots a_j$

$\geq$ no. of ")" in $a_1 \ldots a_j$

$\forall\, j = 1, \ldots, 2n$.

Let C_n be the no. well-formed parentheses of length $2n$.

(4)

Claim:

$$C_n = \sum_{i+j=n-1}^{n} C_i C_j \qquad (*)$$

$$\underline{(()())}()()$$

→ read from left to right

→ note down no. of "(" − no. of ")" until jth stage. δ_j

→ let $2(i+1)$ = min j when $\delta_j = 0$

$$a_1 \cdots a_{2(i+1)}$$

$$= (\underbrace{a_2 \cdots a_{2i+1}}_{\text{well-formed paren of length } 2i}) \ldots$$

$C_0 = 1$

$C_1 = C_0^2 = 1$

$C_2 = C_0C_1 + C_1C_0 = 2$

$C_3 = C_0C_2 + C_1^2 + C_2C_0 = 5$

$C_4 = C_0C_3 + C_1C_2 + C_2C_1 + C_3C_0$
$= 5+2+2+5 = 14$

(*) $\iff f(z) = z f(z)^2 + 1$

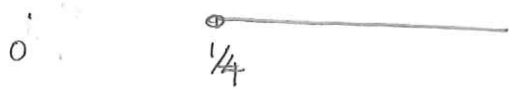
Let $u = f(z)$. get

$$u^2 z - u + 1 = 0$$

Solving gives:

$$u = \frac{1 \pm (1-4z)^{1/2}}{2z}$$

$(1-4z)^{1/2}$ as an analytic fn on $\mathbb{C} - (1/4, \infty)$ unique up to sign.



$$z\left(\sum a_n z^n\right)^2 = \sum b_n z^n$$

$$b_n = \sum_{i+j=n-1} a_i a_j$$

Take the branch that is >0 on $(0, \frac{1}{4})$.

$$\lim_{x \to 0} \frac{1 - \sqrt{1-4x}}{2x}$$

$$= \lim_{x \to 0} \frac{-\frac{1}{2}(-4)(1-4x)^{-1/2}}{2} = 1$$

also $\dfrac{1+(1-4z)^{1/2}}{2z}$ has a pole at 0.

$$f(z) = \frac{1-(1-4z)^{1/2}}{2z}$$

Example: a_n = no. of fixed-point-free involutions in S_n.

$= \{ w \in S_n \mid w^2 = id,\ w(i) \neq i \text{ for any } i \in [n] \}$

$([n] = \{1, \dots, n\})$

$a_n > 0 \iff n$ is even, $n = 2m$

and a_{2m} = no. of permutation of type $\underbrace{(2, \dots, 2)}$

G : finite gp. $n = 2m$.   (7)

$g \in G$.

$$|\text{conj. class of } g| = \frac{|G|}{|Z_G(g)|}$$

$$= \frac{2m!}{2^m m!}$$

$(1\,2)(3\,4)\ldots(2m-1, 2m)$

$$= \frac{1\times 2\times\cdots\times 2m}{\underbrace{2\times 2\times\cdots\times 2}_{m}\ 1\times\cdots\times m}$$

$$= \frac{1\times 2\times\cdots\times 2m}{2\times 4\times\cdots\times 2m}$$

$$= 1\times 3\times\cdots\times(2m-1)$$

$$=: (2m)!! = a_{2m}$$

$a_n = 0$ if n is odd.

(8)

Ex: $\sum \frac{a_n z^n}{n!} = \exp\left(\frac{z^2}{2}\right)$

If $w \in S_n$ has cycle decompo: $(c_1, c_2 \ldots)(c_1' c_2' \ldots)$

$x \in S_n$ xwx^{-1} ~~has cycle decomp~~

$xwx^{-1}(x(c_1)) = x(c_2)$.

xwx^{-1} has cycle decompo.

$(x(c_1), x(c_2)\ldots)(x(c_1'), x(c_2')\ldots)$

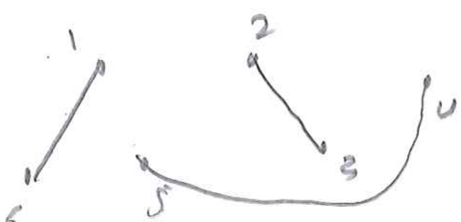
So x commutes with

$(1\,2)(3\,4)\ldots$

$\Leftrightarrow (x(1)\,x(2))(x(3)\,x(4))\ldots$

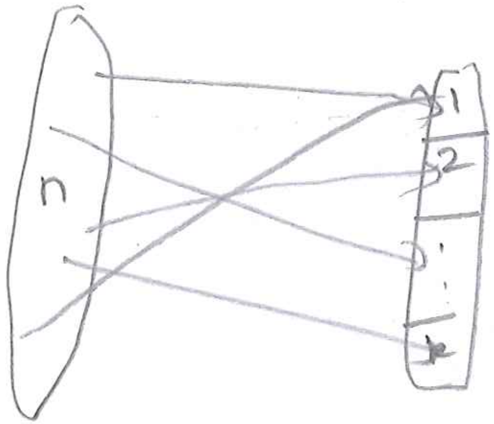
$= (1\,2)(3\,4)\ldots$

$\rightsquigarrow 2^m m!$



Surjection no:

$a_{n,k}$ = no. of surjections $[n] \to [k]$



$$a_{n,k} = \sum_{\substack{n_1+\cdots+n_k=n \\ n_i>0}} \frac{n!}{n_1!\cdots n_k!}$$

$$\sum_{n=0}^{\infty} \frac{a_{n,k} z^n}{n!} = \sum_{n=0}^{\infty} \sum_{\substack{n_1+\cdots+n_k=n \\ n_i>0}} \frac{z^n}{n_1!\cdots n_k!}$$

$$= (e^z-1)^k.$$

Stirling no:

$\left\{ n \atop k \right\}$ = no. of set partitions of $[n]$ into k subsets.

$$S_k(z) = \sum_{n=0}^{\infty} \left\{ n \atop k \right\} \frac{z^n}{n!} = \frac{1}{k!}(e^z-1)^k$$

Bell numbers:

B_n = no. of sets partition of $[n]$
= no. of equivalence relns. on $[n]$

$$\sum \frac{B_n z^n}{n!} = \sum_{k=0}^{\infty} \frac{S_k(z)}{k!}$$

$$= e^{(e^z-1)}$$

<u>Know:</u> $f(z)$ analytic in $B_0(r)$,

$f(z) = \sum_{n=0}^{\infty} a_n z^n$ in a nbd. of 0,

then $\limsup_{n\to\infty} |a_n|^{1/n} \le \frac{1}{r}$.

Radius of convergence

= abs value of singularity nearest to 0

$= \left(\limsup_{n\to\infty} |a_n|^{1/n}\right)^{-1}$

<u>Defn:</u> $\{a_n\}$ has exponential growth of order K if

$\limsup_{n\to\infty} |a_n|^{1/n} = K$

$a_n \bowtie K^n$.

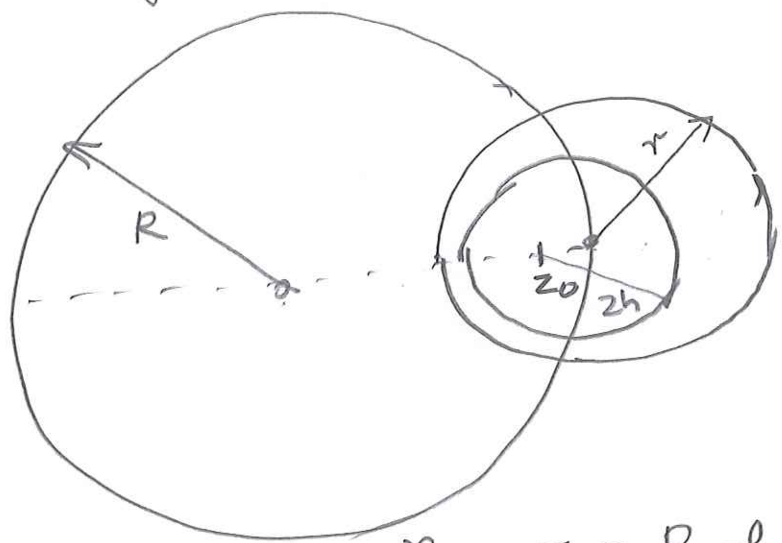
$1/K$ = abs. value of singularity closest to 0.

<u>Pringsheim's thm:</u>

$\{a_n\}$, $a_n \ge 0$ $f(z) = \sum_{n=0}^{\infty} a_n z^n$.

$R = \left(\limsup_{n\to\infty} |a_n|^{1/n}\right)^{-1}$ is the smallest singularity of $f(z)$ on $\mathbb{R}_{\ge 0} \subseteq \mathbb{C}$.

<u>Pf:</u> Suppose R is <u>not</u> a singularity of $f(z)$.



Take $0 < h < \frac{r}{3}$, $z_0 = \underline{R - h}$

(13)

$$\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} b_n (\underbrace{z-z_0}_{u})^n .$$

$z = u + z_0$

$$\sum_{n=0}^{\infty} a_n (u+z_0)^n = \sum_{n=0}^{\infty} b_n u^n$$

Comparing nth derivatives at $u=0$

$$b_n = \sum_{k=n}^{\infty} a_k \binom{k}{n} z_0^{k-n}$$

$f(R+h)$

$$\infty > \sum_{n=0}^{\infty} b_n (2h)^n$$

$$= \sum_{n=0}^{\infty} \sum_{k=n}^{\infty} a_k \binom{k}{n} (R-h)^{k-n} 2h^n$$

$$= \sum_{k=0}^{\infty} a_k \underbrace{\sum_{n=0}^{k} \binom{k}{n} (R-h)^{k-n} 2h^n}_{(R+h)^n}$$

contradiction

(14)

$a_n \bowtie K^n$

$$\limsup_{n\to\infty} \left(\frac{a_n}{K^n}\right)^{1/n} = 1$$

$$\theta(n) = \frac{a_n}{K^n} .$$

$$a_n = \underbrace{K^n}_{\text{exp. growth}} \underbrace{\theta(n)}_{\text{subexp. factor}} .$$