COMPLEX ANALYSIS

HOMEWORK 9

(1) Compute $\int_{-\infty}^{\infty}(1+x^2)^{-4}dx$.

(2) Let $a_n = 1 \times 3 \times \cdots \times (2n-1)$, the product of the first n odd positive integers (which we saw to be the number of fixed-point-free involutions in S_{2n}). Show that
$$\sum_{n=0}^{\infty} \frac{a_n}{n!} z^n = \exp(z^2/2).$$

(3) Let I_n denote the number of involutions in S_n (permutations w with $w^2 = \mathrm{id}$). Show that $I_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} a_k$, where a_k is the number of fixed-point-free involutions in S_{2k}. Using this, and the fact that $\sum_{n=0}^{\infty} a_n z^n/n! = \exp(z^2/2)$, show that
$$\sum_{n=0}^{\infty} \frac{I_n}{n!} z^n = \exp(z + z^2/2).$$

(4) Let $R(x, y)$ be a complex valued rational function of $(x, y) \in \mathbf{R}^2$ which is finite on $\partial B_1(0)$. Let
$$\tilde{R}(z) = z^{-1} R(\tfrac{1}{2}(z + z^{-1}), \tfrac{1}{2i}(z - z^{-1})),$$
show that
$$\int_0^{2\pi} R(\cos\phi, \sin\phi) d\phi = 2\pi \sum_{w \in B_1(0)} \mathrm{res}_w \tilde{R}.$$

(5) Show that
$$\int_0^{2\pi} \frac{d\phi}{a + \sin\phi} = \frac{2\pi}{\sqrt{a^2 - 1}}.$$

Date: Due on 17th October 2018.