COMPLEX ANALYSIS

HOMEWORK 6

(1) Find a fractional linear transformation that takes the half-plane $\mathrm{Re}(z) > 1$ to the half-plane $\mathrm{Im}(z) < 1$.

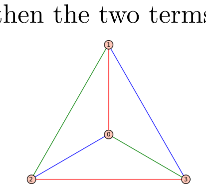
(2) Show that, under the inversion map $z \mapsto 1/z$,
   (a) any line not throught the origin is mapped to a circle through the origin,
   (b) any line through the origin is mapped to a line through the origin,
   (c) any circle not through the origin is mapped to a circle not through the origin, and
   (d) any circle through the origin is mapped to a line not through the origin.

(3) Construct an embedding of the symmetric group S_3 in $PGL_2(\mathbf{C})$ by writing down the fractional linear transformations corresponding to the six permutations of the set $\{0, 1, \infty\} \subset \hat{\mathbf{C}}$.

(4) Let
$$SU(1,1) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \mid a, b, c, d, \in \mathbf{C},\ |a|^2 - |b|^2 = 1 \right\}$$
Show that every element of $SU(1,1)$ is of the form
$$\begin{pmatrix} e^{i\phi} \cosh\theta & e^{i\psi} \sinh\theta \\ e^{-i\psi} \sinh\theta & e^{-i\phi} \cosh\theta \end{pmatrix},$$
for some $\phi, \psi \in [0, 2\pi)$, and $\theta \in \mathbf{R}$.

(5) Let $SL_2(\mathbf{R})$ denote the group of all real 2×2 matrices with determinant 1. Write down an explicit isomorphism $SU(1,1) \to SL_2(\mathbf{R})$ [Hint: use the Cayley transform].

(6) Given two ordered quadruples $\underline{z} = (z_1, z_2, z_3, z_0)$ and $\underline{z}' = (z_1', z_2', z_3', z_0')$ of distinct points in $\hat{\mathbf{C}}$, show that there exists $A \in PGL_2(\mathbf{C})$ taking $\underline{z}$ to $\underline{z}'$ if and only if
$$\frac{(z_1 - z_2)(z_3 - z_0)}{(z_1 - z_0)(z_2 - z_3)} = \frac{(z_1' - z_2')(z_3' - z_0')}{(z_1' - z_0')(z_2' - z_3')}.$$
In other words, the *cross-ratio* map:
$$(z_1, z_2, z_3, z_0) \mapsto \frac{(z_1 - z_2)(z_3 - z_0)}{(z_1 - z_0)(z_2 - z_3)}$$
is a *complete invariant* for the action of $PGL_2(\mathbf{C})$ on $\hat{\mathbf{C}}$. Here if one of z_1, z_2, z_3, or z_4 is infinity, then the two terms containing it are cancelled.



Date: Due on 19th September 2018.