COMPLEX ANALYSIS

HOMEWORK 5

(1) Let $f(z)$ be a holomorphic function on $B_r(0)$ for some r positive, with $f(0) \neq 0$. For a non-zero integer k, show that $z^k f(z)$ is injective on $B_r(0) - \{0\}$ if and only if $k \neq \pm 1$.

(2) Find the isolated singularities in $\mathbf{C}$ of the following functions, and determine whether they are poles, removable, or essential singularities. In case of poles, determine the order:
   (a) $\frac{1-\cos z}{\sin z}$.
   (b) $\frac{z^2-\pi^2}{\sin^2 z}$.
   (c) $\frac{1}{e^z-1}$.
   (d) $\frac{1}{\cos(1/z)}$.

(3) Show that a non-removable singularity of a holomorphic function $f : \Omega - \{c\} \to \mathbf{C}$ at c is always an essential singularity of $\exp \circ f$.

(4) Let $f : \Omega - \{c\} \to \mathbf{C}$ be a holomorphic function, and P be a non-constant polynomial. Show that the singularity of f at c is a removable singularity of f if and only if it is a removable singularity of $P \circ f$, and is a pole of f if and only if it is a pole of $P \circ f$.

(5) Let c_n denote the number of words in the alphabet $\{1, 2\}$ which begin with 1 and are icreasing (all 1's occur before the 2's). Find the generating function $\sum_{n=1}^{\infty} a_n z^n$.

(6) Show that the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, $\mathrm{Re}(s) > 1$, is not a rational function.

(7) Show that the partition generating function $P(n) = \sum_{n=0}^{\infty} p_n z^n$, $|z| < 1$, is not a rational function. Here p_n denotes the number of integer partitions of n.

Date: Due on 10 September 2018.