COMPLEX ANALYSIS

HOMEWORK 4

(1) For all $a, b \in \mathbf{C}$, $|a| < 1 < |b|$, and all positive integers m and n, determine
$$\int_{\partial B_1(0)} \frac{d\zeta}{(\zeta - a)^m (\zeta - b)^n}.$$

(2) Let f be holomorphic in $B_r(0)$, $r > 1$. Calculate the integrals $\int_{\partial B_1(0)} (2 \pm (\zeta + \zeta^{-1}) \frac{f(\zeta)}{\zeta} d\zeta$ in two different ways, and thereby deduce that:
$$\pi^{-1} \int_0^{2\pi} f(e^{it}) \cos^2(t/2) dt = f(0) + \frac{1}{2} f'(0),$$
$$\pi^{-1} \int_0^{2\pi} f(e^{it}) \sin^2(t/2) dt = f(0) - \frac{1}{2} f'(0),$$

(3) Develop into a power series around 0, and determine the radius of convergence of:
$$f(z) = \frac{\sin^2 z}{z}.$$

(4) Determine all holomorphic functions $f : \mathbf{C} \to \mathbf{C}$ which satisfy the functional equation:
$$f(z^2) = (f(z))^2.$$

(5) Suppose that $f : B_1(0) \to \mathbf{C}$ is holomorphic. Show that the diameter $d = \sup_{z,w \in B_1(0)} |f(z) - f(w)|$ of the image of f satisfies:
$$2|f'(0)| \leq d.$$

Hint: Use Cauchy's integral formula to show that $2f'(0) = (2\pi i)^{-1} \int_{\partial B_r(0)} \frac{f(z) - f(-z)}{z^2} dz$.

Date: Due on: 3rd September 2018.