COMPLEX ANALYSIS

HOMEWORK 3

(1) For any polynomial $p(z)$, any $c \in \mathbf{C}$, $r \in \mathbf{R}^+$,
$$\int_{\partial B_r(c)} \overline{p(\xi)} d\xi = 2\pi i r^2 \overline{p'(c)}.$$
(2) Let
$$f(z) = \frac{z}{e^z - 1}.$$
What is the set on which f is holomorphic? For each point $c \in \mathbf{C}$ where f is non holomorphic, compute $\lim_{z \to c}(z - c)f(z)$.
(3) Let $\Omega \subset \mathbf{C}$ be an open set, and $f_n : \Omega \to \mathbf{C}$ be a sequence of functions, all of which have a primitive on Ω. Suppose that $f_n \to f$ uniformly on compact sets. Show that f has a primitive on Ω.
(4) Let G_1 and G_2 be open subsets of $\mathbf{C}$ with $G_1 \cap G_2$ connected. Suppose $f : G_1 \cup G_2 \to \mathbf{C}$ is continuous such that $\int_\gamma f dz = 0$ for every closed path $\gamma \in G_1$ and for every closed path $\gamma \in G_2$. Show that this equality holds for every closed path γ in $G_1 \cup G_2$ as well.
(5) Let $r > 0$, D be an open neighborhood of $\overline{B_r(0)}$, $f : D \to \mathbf{C}$ a holomorphic function, a_1, a_2 distinct points of $B_r(0)$.
   (a) Express $\int_{\partial B_r(0)} \frac{f(\xi)d\xi}{(\xi - a_1)(\xi - a_2)}$ in terms of $\int_{\partial B_r(0)} \frac{f(\xi)d\xi}{\xi - a_i}$, $i = 1, 2$.
   (b) Use the previous part of this exercise to deduce Liouville's theorem: Every bounded holomorphic function on $\mathbf{C}$ is constant.
(6) Let $r > 0$ and $f : \overline{B_r(0)} \to \mathbf{C}$ be a continuous function which is holomorphic on $B_r(0)$. Show that
$$f(z) = \frac{1}{2\pi i} \int_{\partial B_r(0)} \frac{f(\xi)d\xi}{\xi - z} \text{ for all } z \in B_r(0).$$
Note that we are integrating on the *boundary* of a region of holomorphy.

Date: Due on 27th August 2018.