COMPLEX ANALYSIS

HOMEWORK 2

(1) Let $\sigma \in \mathbf{C}$. Define
$$\binom{\sigma}{n} = \frac{\sigma(\sigma-1)\cdots(\sigma-n+1)}{n!}.$$
Show that the binomial series:
$$b_\sigma(z) = \sum_{n=0}^{\infty} \binom{\sigma}{n} z^n$$
satisfies:
$$b'_\sigma(z) = \sigma b_{\sigma-1}(z) = \frac{\sigma}{1+z} b_\sigma(z).$$

(2) The logarithmic series is defined by:
$$\lambda(z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots.$$
Show that $b_\sigma(z) = \exp(\sigma\lambda(z))$.

(3) The sequence $\{C_n\}$ of Catalan numbers is defined by:
$$C_0 = 1, \text{ and } C_{n+1} = C_0C_n + C_1C_{n-1} + \cdots + C_nC_0.$$
Let $f(z) = \sum_{n=0}^{\infty} C_n z^n$. Show that, as a formal power series, $f(z)$ satisfies the recurrence relation $f(x) = 1 + xf(x)^2$.

Solve the above quadratic equation to find an expression of $f(x)$, and hence for C_n. What is the radius of convergence of $f(z)$?

(4) Let π_n denote the number of partitions of n with distinct parts. Show that the generating function satisfies the following formal identity:
$$\sum_{n=0}^{\infty} \pi_n z^n = \prod_{m=1}^{\infty} (1 + z^m).$$
The product on the right hand side should be viewed as the limit of its sequence of partial products, and limits should be computed in the topology of formal power series. What is the radius of convergence of this power series?

(5) Consider a rational function $f(z) = p(z)/q(z)$, where $p(z)$ and $q(z)$ are polynomials with complex coefficients. Suppose $q(z) = (z-c_1)^{m_1}\cdots(z-c_r)^{m_r}$, and has degree greater than that of $p(z)$. Show that $f(z)$ can be expressed in the form:
$$f(z) = \sum_{i=1}^{r} \sum_{j=1}^{m_j} \frac{a_{ij}}{(z-c_i)^j}$$
for some complex numbers a_{ij}.

Date: Due on 20 August 2018.