COMPLEX ANALYSIS

HOMEWORK 12

(1) What is the orbit of i under the action of the subgroup of $PSL_2(\mathbf{R})$ given by matrices of the form:
$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \theta \in \mathbf{R},$$
and under the subgroup of diagonal matrices?

(2) Give a complete set of representatives (exactly one for each orbit) for the orbits of the action of $PSL_2(\mathbf{R})$ on $H \times H$ (with the diagonal action).

(3) Let Γ_1 and Γ_2 be lattices in $\mathbf{C}$. Say that the elliptic curve $\mathbf{C}/\Gamma_1$ is *isomorphic* to $\mathbf{C}/\Gamma_2$ if and only if there exists a biholomorphic map $\phi : \mathbf{C} \to \mathbf{C}$ such that $\phi(\Gamma_1) = \Gamma_2$. For each z in the upper half plane H, let Γ_z be the lattice generated by z and 1 in $\mathbf{C}$. Show that every elliptic curve is isomorphic to a unique elliptic curve of the form $\mathbf{C}/\Gamma_z$, for $z \in PSL_2(\mathbf{Z})\backslash H$.

(4) Let Γ be any lattice in $\mathbf{C}$. Prove that $\sum_{\gamma\in\Gamma-\{0\}} \gamma^{-2k}$ converges absolutely for $k > 1$.

(5) Let G_k denote the Eisenstein series of weight $2k$. Let $g_2 = 60G_2$, and $g_3 = 140G_3$. Using the fact that $G_k(\infty) = 2\zeta(2k)$, and your knowledge of even values of the Riemann zeta function, conclude that $\Delta = g_2^3 - 27g_3^2$ is a cusp form of weight 12.