COMPLEX ANALYSIS

HOMEWORK 1

Final solutions to these problems will be submitted on 13th August. Please prepare to discuss these problems in class on 9th August.

(1) Let $\Omega \subset \mathbf{R}^2$ be an open subset, and $u : \Omega \to \mathbf{R}^2$ be a function for which the partial derivative u_x and u_y exist at every point of Ω and are continuous functions on Ω. Show that u is **C**-differentiable at every point in Ω.

(2) Define the Riemann zeta function by:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Show that ζ is holomorphic at every point s with $\mathrm{Re}(s) > 1$.

(3) Prove the Euler product formula for the Riemann zeta function:

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1} \text{ when } \mathrm{Re}(s) > 1,$$

the product being over all prime numbers.

(4) Let f_n denote the nth Fibonacci number; $f_0 = f_1 = 1$, $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. Determine the radius of convergence of the powoer series

$$f(z) = \sum_{n=0}^{\infty} f_n z^n.$$

(5) Let p_n denote the number of integer partitions of n. In other words p_n is the number of sequences $a_1 \geq a_2 \geq \cdots$ of non-negative integers whose sum is n. Determine the radius of convergence of the power series

$$\sum_{n=0}^{\infty} p_n z^n.$$

(6) Prove Euler's product formula for the partition function:

$$\sum_{n=0}^{\infty} p_n z^n = \prod_{m=1}^{\infty} (1 - z^m)^{-1}.$$

(7) Consider the function

$$u(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0), \\ \frac{x}{y} x^2 + y^2 & \text{otherwise.} \end{cases}$$

Find the partial derivatives of u at all points of $\mathbf{R}^2$. At what points in $\mathbf{R}^2$ is u **R**-differentiable?

Date: Due on: 13th August 2018.