MID-TERM EXAMINATION

ANALYSIS I

(1) Show that the set
$$\{nx - m \mid m, n, \in \mathbf{Z}\}$$
is dense in **R** if and only if x is irrational.

(2) Suppose
$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \text{ for all } |z-a| < R.$$
Show that for any complex number b such that $|b-a| < R$, $f(z)$ has a power series expansion about the point b which is valid for $|z-b| < R - |b-a|$. Conclude that the function $f(z)$ is analytic on $\{z : |z-a| < R\}$.

(3) Find a sequence $\{f_n\}$ of Riemann-integrable real-valued functions on $[0,1]$ which converge point-wise to 0 but for which
$$\lim_{n\to\infty} \int_0^1 f_n(x)dx \neq 0.$$

(4) Let $f : [a,b] \to \mathbf{R}$ be a continuous function taking positive values. Let M denote the maximum value of f on $[a,b]$. Show that
$$\lim_{n\to\infty} \left(\int_a^b |f(x)|^n dx \right)^{\frac{1}{n}} = M.$$

(5) Suppose that the series $\sum a_n$ of positive terms is divergent. Construct a divergent series $\sum b_n$ of positive terms such that $\lim_{n\to\infty}(b_n/a_n) = 0$.

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