HOMEWORK VIII

ANALYSIS I

(1) Show, for all integers $n > 1$, that
$$en^n e^{-n} < n! < en^{n+1}e^{-n}.$$

(2) Suppose that for any real number $k > 0$,
$$f_k(x) = \frac{x^{k-1}e^{-x}}{\Gamma(k)}.$$
Show that for all positive real numbers k and l
$$\int_0^x f_k(y) f_l(x-y) dy = f_{k+l}(x).$$

(3) Show that for all $x, y > 0$, the beta function is given by
$$\mathbf{B}(x,y) = 2\int_0^{\frac{\pi}{2}} \sin^{2x-1}\theta \cos^{2y-1}\theta d\theta.$$

(4) (Apostol, *Mathematical Analysis*, Ex. 13-35) For each real number t define
$$f_t(x) = \begin{cases} xe^{xt}/(e^x - 1) & \text{if } x \in \mathbf{R} \setminus \{0\} \\ 1 & \text{if } x = 0. \end{cases}$$

(a) Show that there exists $\delta > 0$ such that f_t is represented by a power series in x for all $|x| < \delta$.

(b) Define $P_0(t), P_1(t), P_2(t), \ldots$ by the equation
$$f_t(x) = \sum_{n=0}^{\infty} P_n(t)\frac{x^n}{n!}, \text{ for } |x| < \delta.$$
and use the identity
$$\sum_{n=0}^{\infty} P_n(t)\frac{x^n}{n!} = e^{tx}\sum_{n=0}^{\infty} P_n(0)\frac{x^n}{n!}$$
to prove that $P_n(t) = \sum_{k=0}^{n} \binom{n}{k} P_k(0) t^{n-k}$. This shows that each function P_k is a polynomial. P_k is called the kth *Bernoulli polynomial* and $B_k = P_k(0)$ is called the kth *Bernoulli number*.

(c) $B_0 = 1$, $B_1 = -\frac{1}{2}$, $\sum_{k=0}^{n-1} \binom{n}{k} B_k = 0$ if $n = 2, 3, \ldots$.

(d) $P_n'(t) = nP_{n-1}(t)$ for $n = 1, 2, 3, \ldots$.

(e) $P_n(t+1) - P_n(t) = nt^{n-1}$ for $n = 1, 2, 3, \ldots$.

(f) $P_n(1-t) = (-1)^n P_n(t)$ and $B_{2n+1} = 0$ for $n = 1, 2, 3, \ldots$.

(g) $1^n + 2^n + \cdots + (k-1)^n = (P_{n+1}(k) - P_{n+1}(0))/(n+1)$ for $n = 2, 3, \ldots$.

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