HOMEWORK VII

ANALYSIS I

(1) Show that the quadratic polynomial $ax^2 + bx + c$ is convex if and only if $a \geq 0$.
(2) Let $\{a_n\}$ be a sequence of real numbers. Prove or disprove: Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for all $-R < x < R$ for some $R > 0$. Then $f(x)$ is a convex function on $(-R, R)$ if and only if $a_2 \geq 0$.
(3) Show that

$$\sum_{k=1}^{n} \frac{1}{k} = \log n + C + O\left(\frac{1}{n}\right),$$

where C denotes Euler's constant.
(4) Show that

$$\sum_{k=1}^{n} \frac{1}{k^s} = \frac{n^{1-s} - 1}{1 - 2} + C(s) + O\left(\frac{1}{n^s}\right),$$

where

$$C(s) = \lim_{n \to \infty} \left(\sum_{k=1}^{n} \frac{1}{k^s} - \frac{n^{1-s} - 1}{1 - s} \right).$$

In particular the latter limit exists.
(5) If $f(x)$ is convex and continuous, and there exist $a < b < c$ such that $f(a) = f(b) = f(c)$ then $f(x)$ is constant on (a, c).
(6) Suppose $f(x)$ is a convex function on $(0, \infty)$. Let $g(x) = f(x)/x$. Show that either $g(x)$ is monotonic, or the graph of $g(x)$ consists of two monotonic pieces.
(7) If $f(x)$ is a continuous function and every chord (or secant line) of its graph meets the graph in a point distinct from its endpoints then $f(x)$ is a linear function.
(8) If $f(x)$ is convex and $-f(x)$ is convex then $f(x)$ is a linear function.
(9) An increasing convex function of a convex function is convex.
(10) If $f(x)$ is continuous for $x > 0$ and one of $xf(x)$ and $f(1/x)$ is convex then so is the other.

Date: due on 14th October 2005.