HOMEWORK VI

ANALYSIS I

(1) A function $f : \mathbf{N} \to \mathbf{C}$ is called *multiplicative* if for any positive integers m and n whose greatest common divisor is one, $f(mn) = f(m)f(n)$ and *completely multiplicative* if for any positive integers m and n, $f(mn) = f(m)f(n)$. Prove the following theorem, discovered by Euler in 1737:
Let $f : \mathbf{N} \to \mathbf{C}$ be a multiplicative function such that the series $\sum f(n)$ is absolutely convergent. Then the sum of the series can be expressed as an absolutely convergent product:

$$\sum_{n=1}^{\infty} f(n) = \prod_{p} \{1 + f(p) + f(p^2) + \cdots\}$$

where the product ranges over all prime numbers p. If f is completely multiplicative then the product simplifies and

$$\sum_{n=1}^{\infty} f(n) = \prod_{p} \frac{1}{1 - f(p)}.$$

(2) Prove the Euler product expansion for the Riemann zeta function:

$$\prod_{p \text{ prime}} \left(\frac{1}{1 - p^{-s}} \right) = \sum_{n=1}^{\infty} \frac{1}{n^s} \text{ for all } s > 1.$$

(3) Show that the infinite product over all primes $\prod_p (\frac{1}{1-p^{-1}})$ does not converge.

(4) Let $\{a_n\}$ be a sequence of positive numbers. Show that if $\prod_{n=1}^{\infty}(1 + a_n)$ converges, then so does $\sum_{n=1}^{\infty} \log(1 + a_n)$ (the converse was proved in class).

(5) Consider the function $f : [0, 1] \to \mathbf{R}$ given by

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q}, \quad p, q \in \mathbf{Z}, \quad \gcd(p, q) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Show that the set of points where f is not continuous is of measure zero.
   (b) Show that f is Riemann integrable (without invoking the Riemann-Lebesgue theorem).

(6) Let $f : [a, b] \to \mathbf{R}$ be a Riemann integrable function. Prove that for any $\epsilon > 0$ there exists $\delta > 0$ such that whenever $P = (a = x_0 \leq \cdots \leq x_n = b)$ is a partition of $[a, b]$ such that $x_i - x_{i-1} < \delta$ and $x'_1, \ldots, x'_n$ are such that $x_{i-1} \leq x'_i \leq x_i$ for $1 \leq i \leq n$, then

$$\left| \sum_{i=1}^{n} (x_i - x_{i-1}) f(x'_i) - \int_a^b f(x)dx \right| < \epsilon.$$

(7) Show that

$$\lim_{n\to\infty} \frac{1 + \cos\frac{x}{n} + \cos\frac{2x}{n} + \cdots + \cos\frac{(n-1)x}{n}}{n} = \frac{\sin x}{x}.$$

(8) If $f : [a, b] \to \mathbf{R}$ is continuous and $\phi : [a, b] \to \mathbf{R}$ takes only non-negative values, show that there exists $a \leq \xi \leq b$ such that

$$\int_a^b f(x)\phi(x)dx = f(\xi) \int_a^b \phi(x)dx.$$

Date: due on 23rd September 2005.