HOMEWORK V

ANALYSIS I

(1) Let $F(a, b, c; z)$ denote the hypergeometric function with parameters a, b, c. Show that
$$e^z = \lim_{\beta\to\infty} F(1, \beta, 1; z/\beta).$$

(2) Show that for complex numbers a, b, c,
$$\left|\frac{(a+n-1)(b+n-1)}{n(c+n-1)}\right| = \left|1 + \frac{a+b-c-1}{n} + O\left(\frac{1}{n^2}\right)\right|.$$

(3) Show that for real numbers $a', b', c', a'', b'', c''$,
$$\left|\left(1 + \frac{a'+b'-c'-1}{n}\right) + i\left(\frac{a''+b''-c''}{n}\right) + O\left(\frac{1}{n^2}\right)\right| = 1 + \frac{a'+b'-c'-1}{n} + O\left(\frac{1}{n^2}\right).$$

(4) Show that for all complex numbers $|z| \le 1$, the function $f(z)$ given by the power series
$$f(z) = 1 + \frac{1}{2}z + \frac{1}{2!}\frac{1}{2}\left(\frac{1}{2} - 1\right)z^2 + \frac{1}{3!}\frac{1}{2}\left(\frac{1}{2} - 1\right)\left(\frac{1}{2} - 2\right)z^2 + \cdots$$
satisfies $f(z)^2 = 1 + z$.

(5) Prove that
$$\sum_{(m_1,\ldots,m_r)\in\mathbf{Z}^r-\{(0,\ldots,0)\}} \frac{1}{(m_1^2 + \cdots + m_r^2)^\mu}$$
is absolutely convergent if $\mu > \frac{1}{2}r$. (In the case of a positive series, the multiple sum exists if and only if any iterated sum converges, so you may work with an iterated sum in this problem).

(6) Show that for $t > 0$
$$\frac{1}{t^2} = \frac{1}{t(t+1)} + \frac{1}{t(t+1)(t+2)} + \frac{1\cdot 2}{t(t+1)(t+2)(t+3)} + \cdots$$

(7) (Stirling) Use the previous exercise to convert
$$\frac{1}{t^2} + \frac{1}{(t+1)^2} + \frac{1}{(t+2)^2} + \cdots$$
into a double series, and transform it to
$$\frac{1}{t} + \frac{1}{2t(t+1)} + \frac{1\cdot 2}{3t(t+1)(t+2)} + \frac{1\cdot 2\cdot 3}{4t(t+1)(t+2)(t+3)} + \cdot$$
Take $t = 10$ and so calculate $\zeta(2) = \sum \frac{1}{n^2}$ to seven decimal places (of course, you may use a calculator for this).

Date: due on 12th September 2005.