HOMEWORK IV

ANALYSIS I

(1) Show that $\mathbf{R}$ is a connected topological space.
(2) Suppose S is any subset of $\mathbf{R}$ such that there exist real numbers $x < y < z$ such that $x, z \in S$ and $y \notin S$. Show that S is not connected. Thus the only connected subsets of $\mathbf{R}$ are intervals (this includes open, closed, half open and infinite intervals).
(3) Show that $1/(z - a)$ is an analytic function on $\mathbf{C} \setminus \{a\}$.
(4) Prove or give a counterexample: given any array $\{a_{m,n}\}$ of complex numbers, the double series $\sum a_{m,n}$ converges if and only if the iterated series

$$\sum_{k=1}^{\infty} \sum_{i=1}^{k} a_{i,k-i}$$

converges, and then the two sums are equal.
(5) Show that if $\sum a_{m,n}$ converges absolutely (i.e., the double series of absolute values converges), then the sums by rows and sums by columns exist, and are equal.
(6) (Apostol's *Mathematical Analysis*, p.424, Ex. 13-6) Let $\{f_n\}$ be a sequence of continuous functions defined on a compact set S and assume that $\{f_n\}$ converges point-wise on S to a limit function f. Show that $f_n \to f$ uniformly on S, if and only if, the following two conditions hold:
   (a) The limit function f is continuous on X.
   (b) For every $\epsilon > 0$, there exists an $m > 0$ and a $\delta > 0$ such that $n > m$ and $|f_k(x) - f(x)| < \delta$ implies $|f_{k+n}(x) - f(x)| < \epsilon$ for all $x \in S$ and all $k = 1, 2, \ldots$.

   Hint: To prove the sufficiency of (a) and (b), show that for each x_0 in S there exists $\theta > 0$ and an integer k (depending on x_0) such that

$$|f_k(x) - f(x)| < \delta \text{ if } |x - x_0| < \theta.$$

   By compactness, a finite set of integers, say $A = \{k_1, \ldots, k_r\}$, has the property that, for each $x \in S$, some k in A satisfies $|f_k(x) - f(x)| < \delta$. Uniform convergence is an easy consequence of this fact.
(7) (Due to Dini, see loc. cit. Ex. 13-7) Use the previous exercise to prove that it $\{f_n\}$ is a sequence of real-valued functions converging point-wise to a continuous limit f on a compact set S, and if $f_n(x) \geq f_{n+1}(x)$ for each $x \in S$ and every $n \in \mathbf{N}$, then $f_n \to f$ uniformly on S.
(8) Let $f_n(x) = 1/(nx+1)$ for $x \in (0, 1)$, $n \in \mathbf{N}$. Show that $\{f_n\}$ converges point-wise but not uniformly on $(0, 1)$.
   [This example demonstrates that the compactness of S is essential in the previous exercise]
(9) Let $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ be two convergent series and let $c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$. If $\sum_{n=0}^{\infty} c_n$ converges then show that

$$\sum_{n=0}^{\infty} c_n = \left(\sum_{n=0}^{\infty} a_n \right) \left(\sum_{n=0}^{\infty} b_n \right).$$

   [*Hint:* Use the theorem on products of power series to show that $\sum_{n=0}^{\infty} c_n x^n$ converges for $x \in (-1, 1)$. Then use Abel's on continuity up to the circle of convergence.]
(10) Assume that $f(x) = \sum a_n x^n$ converges for $|x| < 1$. If each $a_n \geq 0$ and if $\sum a_n$ diverges, show that

$$\lim_{x \to 1-} f(x) = \infty.$$

Date: due on 5th September 2005.