HOMEWORK III

ANALYSIS I

(1) Show that the following set K is compact:

$$K = \left\{ \frac{1}{n} \;\middle|\; n \in \mathbf{N} \right\} \cup \{0\}.$$

(2) Let E be the set of real numbers between 0 and 1 whose decimal expansion contains only the digits 4 and 7. Is E countable? Is it dense in $[0,1]$? Is it compact?

(3) Investigate the convergence of the series

$$\sum_{n=1}^{\infty} \left\{ \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots 2n} \cdot \frac{4n+3}{2n+2} \right\}^2 .$$

(4) Show that when $s > 1$,

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{s-1} + \sum_{n=1}^{\infty} \left[\frac{1}{n^s} + \frac{1}{s-1} \left\{ \frac{1}{(n+1)^{s-1}} - \frac{1}{n^{s-1}} \right\} \right].$$

and show that the series on the right converges when $0 < s < 1$.

(5) If

$$a_{m,n} = \frac{(m-n)}{2^{m+n}} \frac{(m+n-1)!}{m!\, n!}, \text{ for } m, n > 0,$$
$$a_{m,0} = 2^{-m}, \; a_{0,n} = -2^{-n}, \text{ and } a_{0,0} = 0$$

show that

$$\sum_{m=0}^{\infty} \left(\sum_{n=0}^{\infty} a_{m,n} \right) = -1, \; \sum_{n=0}^{\infty} \left(\sum_{m=0}^{\infty} a_{m,n} \right) = 1.$$

(6) Find a countable family of open sets in $\mathbf{R}$ such that every open subset of $\mathbf{R}$ can be written as a union of sets in this family. This is known as the *second countability property* in topology.

(7) Let X be the space of all bounded functions $f : [0,1] \to \mathbf{C}$. For $f, g \in X$ define

$$d(f,g) = \sup\{|f(x) - g(x)| \;:\; x \in [0,1]\}.$$

Show that d gives X the structure of a metric space.

(8) Does X from the previous problem have a countable dense subset?

(9) Does there exist an uncountable family $\{U_\alpha\}$ of subsets of $\mathbf{N}$ such that any two sets in the family (a) are disjoint, (b) have only a finite number of elements in their intersection?

(10) If K_1 and K_2 are compact subsets of $\mathbf{R}$ then

$$K = \{x + iy \mid x \in K_1 \text{ and } y \in K_2\} \subset \mathbf{C}$$

is a compact set.

Date: due on 26th August 2005.