HOMEWORK II

ANALYSIS I

(1) Use Cauchy's condensation test to show that the series
$$\sum_{n=1}^{\infty} \frac{1}{n \log n}$$
is convergent.

(2) Suppose the series $\sum 1/C_n$ (of positive terms) is convergent. Then the series $\sum a_n$ (also consisting of positive terms) converges if the sequence $\{a_n C_n\}$ is bounded.

(3) Suppose the series $\sum 1/D_n$ (of positive terms) is divergent. Then the series $\sum a_n$ (also consisting of positive terms) diverges if the sequence $\liminf a_n C_n > 0$.

(4) Show that
$$\liminf(a_{n+1}/a_n) \leq \liminf a_n^{1/n} \text{ and } \limsup a_n^{1/n} \leq \limsup(a_{n+1}/a_n).$$

(5) Use Exercise 4 and Cauchy's root test to prove D'Alembert's ratio test, which states that if $\liminf(a_n/a_{n+1}) > 1$ then $\sum a_n$ diverges, and if $\limsup(a_n/a_{n+1}) < 1$ then $\sum a_n$ converges.

(6) A subset U of $\mathbf{R}$ is said to be *open* if for every element $x \in S$, there exists a positive real number ϵ such that $(x - \epsilon, x + \epsilon) \subset U$. Verify that this defines a topology on $\mathbf{R}$.

(7) Show that the real numbers form a Hausdorff topological space.

(8) For $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{R}^n$ define
$$|\mathbf{x}| = \sqrt{x_1^2 + \cdots + x_n^2}.$$
Definition $d(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|$ for $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$. Show that the above definition gives $\mathbf{R}^n$ the structure of a metric space.

Date: due on 19 August 2005.