HOMEWORK X

ANALYSIS I

(1) Suppose $0 < \delta < \pi$, $f(x) = 1$ if $|x| < \delta$, $f(x) = 0$ if $\delta < x \leq |\pi|$ and f is 2π-periodic,
   (a) Compute the Fourier coefficients of f.
   (b) Conclude that
$$\sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n} = \frac{\pi - \delta}{2}.$$
   (c) Deduce from Parseval's theorem that
$$\sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{n^2\delta} = \frac{\pi - \delta}{2}.$$
   (d) Let $\delta \to 0$ and prove that
$$\int_0^{\infty} \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}.$$
   (e) Put $\delta = \frac{\pi}{2}$ in (1c). What do you get?

(2) Let f be the 2π-periodic function for which $f(x) = x$ for $0 \leq x < 2\pi$. Apply Parseval's theorem to compute
$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

(3) Let f be the 2π-periodic function for which $f(x) = x^2$ for $0 \leq x < 2\pi$. Apply Parseval's theorem (and use the previous excercise) to compute
$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4}.$$

(4) (optional) What can you say about $\zeta(2k)$ for $k = 1, 2, 3, \ldots$?

Date: due on 18th November 2005.