HOMEWORK I

ANALYSIS I

(1) (a) Define addition and mutliplication of Dedekind cuts.
    (b) Verify that the operations you define extend the corresponding operations on the real numbers.
    (c) Let $\sqrt{2}$ be the Dedekind cut for which the left subset is

$$\{x \in \mathbf{Q} | x^2 < 2 \text{ or } x < 0\}.$$

Using your definition of multiplication of Dedekind cuts show that $(\sqrt{2})^2 = 2$.

(2) Show that if $p_0 = p_n = 1$ and neither of

$$1 + p_1 + p_2 + \cdots \quad \text{and} \quad 1 - p_1 + p_2 - \cdots$$

is zero then the equation in the previous problem can have no rational root.

(3) Find the rational roots (if any) of

$$x^4 - 4x^3 - 8x^2 + 13x + 10 = 0.$$

(4) If a and b are rational then $\sqrt{a} + \sqrt{b}$ can not be rational unless $\sqrt{a}$ and $\sqrt{b}$ are rational.

(5) Show that $\sqrt{2} + \sqrt{3}$ is algebraic.

(6) A line AB is divided at C in such a way that $AB \cdot AC = BC^2$. Show that the ratio AC/AB is irrational.

(7) The number $10^{-1!} + 10^{-2!} + 10^{-3!} + \cdots$ is transcendental.

(8) For any irrational number x show that there exist infinitely many fractions p/q $(q > 0)$ such that $|x - p/q| < 1/q^2$. [Hint: for each $q > 1$, the farctional parts of two of the $q + 1$ numbers $0, x, \ldots, (q-1)x$ differ by less than $1/q$.]

(9) For which real numbers does the sequence $\sin(n\theta\pi)$ converge? For which real numbers does it not converge? Give complete proofs in each case.

(10) Prove that $\sin(n!\theta\pi)$ converges to 0 for all rational values of θ. What can you say about this sequence when θ is irrational?

Date: due on 12 August 2005.