FINAL EXAMINATION

ANALYSIS I

(1) (a) Prove that the series

$$X = \frac{1}{x} + \sum_{n \in \mathbf{Z} \setminus \{0\}} \left(\frac{1}{n} + \frac{1}{x-n} \right)$$

converges whenever x is not an integer.

(b) Show that

$$\lim \left(\sum_{-p}^{q} \frac{1}{x-n} \right) - X = -\log k,$$

where p and q tend to ∞ in such a way that $\lim(q/p) = k$.

(2) For continuous 2π-periodic functions $f, g : \mathbf{R} \to \mathbf{C}$ let $f * g$ be the function given by

$$(f * g)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) g(t) dt.$$

Show that $f * g$ is 2π-periodic and continuous, and that the Fourier coefficients of $f * g$ are given by

$$c_n(f * g) = c_n(f) c_n(g) \text{ for all } n \in \mathbf{Z}.$$

(3) (a) Show that

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} = \int_0^1 \frac{1 - (1-t)^n}{t} dt.$$

(b) Show that Euler's constant[1] γ is given by

$$\lim_{n \to \infty} \left[\int_0^1 \left\{ 1 - \left(1 - \frac{t}{n} \right)^n \right\} \frac{dt}{t} - \int_1^n \left(1 - \frac{t}{n} \right)^n \frac{dt}{t} \right]$$

(4) (a) Let

$$u_n = \frac{(n + a_1)(n + a_2) \cdots (n + a_k)}{(n + b_1)(n + b_2) \cdots (n + b_l)}.$$

Show that if the infinite product $\prod_{n=1}^{\infty} u_n$ converges, then $k = l$ and $a_1 + \cdots + a_k = b_1 + \cdots + b_l$.

(b) When these conditions are satisfied[2], show that

$$\prod_{n=1}^{\infty} u_n = \frac{\Gamma(1 + b_1)\Gamma(1 + b_2) \cdots \Gamma(1 + b_l)}{\Gamma(1 + a_1)\Gamma(1 + a_2) \cdots \Gamma(1 + a_k)}.$$

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[1] Recall that $\gamma = \lim_{m \to \infty} \{1 + 2^{-1} + \cdots + m^{-1} - \log m\}$.

[2] Recall that $\Gamma(x) = \lim_{n \to \infty} \frac{n^{x-1} n!}{x(1+x) \cdots (n-1+x)}$.