HOMEWORK IX

ANALYSIS I

(1) Show that every monotone function $f : [a, b] \to \mathbf{R}$ is Riemann integrable.

(2) Show that
$$\int_0^\infty \frac{\sin(x^3 - ax)}{x}$$
converges.

(3) Define
$$f(x) = \left(\int_0^x e^{-t^2} dt\right)^2, \quad g(x) = \int_0^1 \frac{e^{-x^2(t^2+1)}}{t^2+1} dt.$$
(a) Show that $g'(x) + f'(x) = 0$ for all x and deduce that $g(x) + f(x) = \pi/4$.
(b) Use (a) to prove that
$$\int_0^\infty e^{-t^2} dt = \frac{1}{2}\sqrt{\pi}.$$

(4) Suppose g is Riemann integrable on $[a, b]$ and define $f(x) = \int_a^x g(t)dt$ if $x \in [a, b]$. Show that $\int_a^x |g(t)|dt$ gives the total variation of f on $[a, x]$.

(5) Let f be a positive continuous function on $[a, b]$. Let M denote the maximum value of f on $[a, b]$. Show that
$$\lim_{n\to\infty} \left[\int_a^b f(x)^n dx\right]^{1/n} = M.$$

Date: due on 2 November 2004.