HOMEWORK VIII

ANALYSIS I

(1) Show that the integral $\int_0^\infty \sin(x^2)dx$ converges.

(2) If a is real, show that
$$\int_0^\infty \frac{\cos(ax)}{1+x^2}dx$$
is a continuous function of a.

(3) If on an interval $[a,b]$ the function $f(x)$ is continuous and ϕ a Riemann integrable function such that $\phi(x) \geq 0$ for all x show that there exists $\xi \in [a,b]$ such that
$$\int_a^b f(x)\phi(x)dx = f(\xi)\int_a^b \phi(x)dx.$$
Construct an example to show that the hypothesis $\phi(x) \geq 0$ is necessary.

(4) By writing $|\phi(x) - \phi(b)|$ in place of $\phi(x)$ in Bonnet's version of the mean value theorem show that if $\phi(x)$ is a monotonic function, then a number ξ exists such that $a \leq \xi \leq b$ and
$$\int_a^b f(x)\phi(x)dx = \phi(a)\int_a^\xi f(x)dx + \phi(b)\int_\xi^b f(x)dx.$$

(5) Show that $\int_1^\infty \frac{\sin x}{x}dx$ converges. What about $\int_0^\infty \frac{\sin x}{x}dx$?

(6) Recall that γ denotes *Euler's constant* [Homework VII, (1) and (4)]. Show that
$$\gamma = \lim_{n\to\infty}\left[\int_0^1 \left\{1 - \left(1 - \frac{t}{n}\right)^n\right\}\frac{dt}{t} - \int_1^n \left(1 - \frac{t}{n}\right)^n \frac{dt}{t}\right].$$
[Hint: First show that $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \int_0^1 \frac{1-(1-t)^n}{t}dt$.]

(7) Show (justifying all the steps) that
$$\gamma = \int_0^1 \frac{1-e^{-t}}{t}dt - \int_1^\infty \frac{e^{-t}}{t}dt.$$

Date: due on 19 October 2004.