HOMEWORK VII

ANALYSIS I

(1) Show that the limit
$$\lim_{m\to\infty}\left\{\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{m}-\log m\right\}$$
exists. [Hint: $\log m = \int_1^m \frac{1}{t}dt$. Now break up the interval $[1, m]$ into m equal parts and compare the integral on the nth part with $\frac{1}{n}$ to show that the given sequence is increasing and bounded above.]

(2) Let
$$S_n(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots + \frac{(-1)^{n-1}}{n}x^n.$$
Show that for all $x > 0$,
$$0 < S_n(x) - \log(1+x) < \frac{1}{n+1}x^{n+1} \quad \text{if} \quad n \text{ is odd},$$
$$0 < \log(1+x) - S_n(x) < \frac{1}{n+1}x^{n+1} \quad \text{if} \quad n \text{ is even}.$$
If $-1 < x < 0$, show that
$$0 < S_n(x) - \log(1+x) < \frac{|x|^{n+1}}{(n+1)(1+x)} \quad \text{if} \quad n \text{ is odd},$$
$$0 < \log(1+x) - S_n(x) < \frac{|x|^{n+1}}{(n+1)(1+x)} \quad \text{if} \quad n \text{ is even}.$$
[Hint: $\log(1+x) = \int_0^x \frac{1}{1+t}$. Expand the integrand to n terms plus a remainder and bound the integral of the remainder.]

(3) Let
$$P_n = \left(1+\frac{x}{1}\right)\left(1+\frac{x}{2}\right)\left(1+\frac{x}{3}\right)\cdots\left(1+\frac{x}{n}\right)$$
Show that the sequence $T_n = x(1 + 1/2 + 1/3 + \cdots + 1/n) - \log P_n$ is convergent. [Hint: use the inequality from (2) with $n = 1$.]

(4) Let $T = \lim_{n\to\infty} T_n$ (from the previous problem). Show that, for all $x > -1$,
$$\lim_{n\to\infty}\frac{(1+x)(2+x)\cdots(n+x)}{n^x n!} = e^{\gamma x - T} = e^{\gamma x}\prod_{r=1}^{\infty}\left(1+\frac{x}{r}\right)e^{-\frac{x}{r}},$$
where γ is the limit from (1).

(5) Show that, for all $x > -1$,
$$e^{\gamma x}\prod_{r=1}^{\infty}\left(1+\frac{x}{r}\right)e^{-\frac{x}{r}} = \frac{1}{\Gamma(x)}$$

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