HOMEWORK VI

ANALYSIS I

(1) A function $f : \mathbf{N} \to \mathbf{C}$ is called *multiplicative* if for any positive integers m and n whose greatest common divisor is one, $f(mn) = f(m)f(n)$ and *completely multiplicative* if for any positive integers m and n, $f(mn) = f(m)f(n)$. Prove the following theorem, discovered by Euler in 1737:
Let $f : \mathbf{N} \to \mathbf{C}$ be a multiplicative function such that the series $\sum f(n)$ is absolutely convergent. Then the sum of the series can be expressed as an absolutely convergent product:
$$\sum_{n=1}^{\infty} f(n) = \prod_{p} \{1 + f(p) + f(p^2) + \cdots\}$$
where the product ranges over all prime numbers p. If f is completely multiplicative then the product simplifies and
$$\sum_{n=1}^{\infty} f(n) = \prod_{p} \frac{1}{1 - f(p)}.$$

(2) Show that the infinite product over all primes $\prod_p (\frac{1}{1-p^{-1}})$ does not converge.

(3) Let $\{a_n\}$ be a sequence of positive numbers. Show that if $\prod_{n=1}^{\infty}(1 + a_n)$ converges, then so does $\sum_{n=1}^{\infty} \log(1+a_n)$ (the converse was proved in class).

(4) Since $\cos(\frac{1}{2}\pi) = 0$, it must be that $\sin(\frac{1}{2}\pi) = \pm 1$. Using only the definitions and results we have proved in class show that $\sin(\frac{1}{2}\pi) = +1$, not -1.

(5) Verify the identity
$$\left(1 - \frac{x}{1}\right)\left(1 - \frac{x}{2}\right)\cdots\left(1 - \frac{x}{n}\right)$$
$$= 1 - x + \frac{x(x-1)}{2!} - \frac{x(x-1)(x-2)}{3!} + \cdots + (-1)^n \frac{x(x-1)\cdot(x-n+1)}{n!}$$
Show that as n tends to ∞, the product diverges for all values of x except 0, but the series converges, provided that $x > 0$.

(6) Prove that $(1+x)(1+x^2)(1+x^4)(1+x^8)\cdots = 1/(1-x)$, if $|x| < 1$.

Date: due on 21 September 2004.