HOMEWORK V

ANALYSIS I

(1) Let $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ be two convergent series and let $c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$. If $\sum_{n=0}^{\infty} c_n$ converges then show that

$$\sum_{n=0}^{\infty} c_n = \left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right).$$

[*Hint:* Use the theorem on products of power series to show that $\sum_{n=0}^{\infty} c_n x^n$ converges for $x \in (-1, 1)$. Then use Abel's on continuity up to the circle of convergence.]

(2) Assume that $f(x) = \sum a_n x^n$ converges for $|x| < 1$. If each $a_n \geq 0$ and if $\sum a_n$ diverges, show that

$$\lim_{x \to 1-} f(x) = \infty.$$

(3) Show that for $t > 0$

$$\frac{1}{t^2} = \frac{1}{t(t+1)} + \frac{1}{t(t+1)(t+2)} + \frac{1 \cdot 2}{t(t+1)(t+2)(t+3)} + \cdots$$

(4) (Stirling) Use the previous exercise to convert

$$\frac{1}{t^2} + \frac{1}{(t+1)^2} + \frac{1}{(t+2)^2} + \cdots$$

into a double series, and transform it to

$$\frac{1}{t} + \frac{1}{2t(t+1)} + \frac{1 \cdot 2}{3t(t+1)(t+2)} + \frac{1 \cdot 2 \cdot 3}{4t(t+1)(t+2)(t+3)} + \cdot$$

Take $t = 10$ and so calculate $\zeta(2) = \sum \frac{1}{n^2}$ to seven decimal places.

(5) Show that
   (a) $\sum_{m+1}^{m+p} \cos(n\theta) = \sin(p\theta/2) \cos[\{m + \frac{1}{2}(p+1)\}\theta] \operatorname{cosec}(\theta/2)$.
   (b) $\sum_{m+1}^{m+p} \sin(n\theta) = \sin(p\theta/2) \sin[\{m + \frac{1}{2}(p+1)\}\theta] \operatorname{cosec}(\theta/2)$.

(6) Suppose $u_n \geq u_{n+1} \geq 0$ and $\lim_{n\to\infty} u_n = 0$. Show that the series $\sum_{n=1}^{\infty} u_n \cos(nx)$ and $\sum_{n=1}^{\infty} u_n \sin(nx)$ converge for all real values of x other than 0 and multiples of 2π. Show that the convergence is uniform on a closed interval $[a, b]$ where $0 < a < b < 2\pi$. [Hint: Use the Dirichlet type test for uniform convergence from last week's assignment.]

Date: due on 14 September 2004.